FINITE-TIME SINGULARITY FORMATION FOR THE THREE-DIMENSIONAL EULER EQUATIONS

LEVENT ALPÖGE, TRISTAN BUCKMASTER, AND MATEI P. COICULESCU

Abstract. We construct smooth periodic initial data for the three-dimensional incompressible Euler equations whose solution blows up in finite time. The velocity remains bounded, while the vorticity magnitude at the origin tends to infinity. Away from the origin, the velocity and its time derivative have smooth spatial limits at the singular time. The construction uses localized Craik–Criminale waves as building blocks of a multi-scale construction.

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1. Introduction

The incompressible Euler equations are among the earliest partial differential equations derived in mathematical physics. Euler presented them in 1755, less than a decade after d'Alembert derived the wave equation for the vibrating string [41, 42].

The goal of this work is to construct smooth initial data for the incompressible Euler equations on the three-dimensional torus whose solution blows up in finite time. We represent $\mathbb{T}^3$ by the cube $[-1/2, 1/2]^3$ with periodic boundary conditions (opposite faces identified). We write these equations as

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \qquad \operatorname{div} u = 0 \quad \text{on } \mathbb{T}^3 \times [0, T_*). \tag{1.1}$$

Local existence and uniqueness for sufficiently regular data go back to Lichtenstein and Günther in the 1920s [43, 44]. Classical theory on $\mathbb{T}^3$ supplies a unique maximal smooth solution from smooth divergence-free data [13, 10]. By the Beale–Kato–Majda criterion [9, 11], a smooth solution on $[0, T)$ continues smoothly through $T < \infty$ if $\int_0^T \|\operatorname{curl} u(t)\|_{L^\infty}\, \mathrm{d}t < \infty$.

In two dimensions the scalar vorticity is transported, so its L^∞ norm is conserved, and smooth solutions exist for all time [45]. Yudovich [46] proved global existence and uniqueness for bounded,

possibly nonsmooth initial vorticity. In three dimensions, Fefferman's description of the Navier–Stokes Millennium Problem stated that the global regularity question was also open for the Euler equations on $\mathbb{R}^3$ and on $\mathbb{T}^3$ [47].

Numerical searches for blow-up include the Taylor–Green vortex study of Morf, Orszag and Frisch [48] and Kerr's computations for antiparallel vortex tubes [49]. In a later computation, Hou and Li [50] observed at most double-exponential growth of the maximum vorticity past Kerr's predicted singular time. Luo and Hou [51] found numerical evidence for axisymmetric blow-up at the wall of an axially periodic cylinder. Childress, Gilbert and Valiant [52] studied, analytically and numerically, eroding vortex dipoles in axisymmetric flow without swirl, whose maximal vorticity grows like $t^{4/3}$, and Childress and Gilbert [53] proposed, by formal asymptotic analysis, the hairpin geometry as a model for finite-time blow-up.

On $\mathbb{R}^3$, Elgindi [15] constructed finite-time blow-up from C^{1,α_0} initial data for some $\alpha_0 > 0$. Elgindi, Ghoul and Masmoudi [16] obtained stable, locally self-similar blow-up for finite-energy data in this regularity class. Chen and Hou [28, 29] prove nearly self-similar blow-up from smooth axisymmetric data in a periodic cylinder with boundary, using computer-assisted estimates. The theorem below concerns smooth periodic data on $\mathbb{T}^3$, with bounded velocity and vorticity growth concentrated at the origin.

Theorem 1.1. *There is an odd divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ whose maximal classical solution u of (1.1) is smooth and odd on $\mathbb{T}^3 \times [0,T_*)$ for a finite $T_* > 0$, and satisfies*

$$\int_0^{T_*} \|\nabla u(t)\|_{L^\infty}\,\mathrm{d}t = \infty, \qquad \int_0^{T_*} \|\operatorname{curl} u(t)\|_{L^\infty}\,\mathrm{d}t = \infty. \tag{1.2}$$

Moreover,

$$\sup_{0\le t<T_*} \|u(t)\|_{L^\infty} < \infty, \qquad \lim_{t\uparrow T_*} |\operatorname{curl} u(t,0)| = \infty. \tag{1.3}$$

For every compact $K \subset \mathbb{T}^3 \setminus \{0\}$ and every integer $m \ge 0$, both $u(t)$ and $\partial_t u(t)$ converge in $C^m(K)$ as $t \uparrow T_*$.

Remark 1.2. *The solution in Theorem 1.1 may be chosen so that it extends smoothly in space and time through T_* away from the origin. A proof is given in the auxiliary note* [31].

We construct a sequence of smooth approximate solutions of the Euler equations whose residuals tend to zero. The sequence converges in every spatial C^m norm, uniformly on compact time intervals before T_*, to a solution of the Euler equations.

At each step in the sequence we add an oscillation at a higher frequency, localized near the origin and initially small in velocity. The velocity gradient of the preceding scales amplifies the new oscillation, whose gradient in turn drives the growth of the next one. The growth intervals have finite total length. This multiscale construction follows the approach of Córdoba and Martínez-Zoroa for the Euler equations with forcing and for forced IPM [18, 19]. Córdoba, Martínez-Zoroa and Zheng [17] used a cascade of vorticity layers to obtain blow-up for the unforced Euler equations on $\mathbb{R}^3$ from $C^{1,\alpha}$ data smooth away from the origin. Related constructions give blow-up with forcing for the inviscid Boussinesq equations, the variable-density Euler equations and the hypodissipative Navier–Stokes equations [20, 21, 22]. Our companion papers proving singularity formation for forced IPM, the forced Boussinesq equations, and the forced Euler equations [3, 1, 2] use forces smooth in space and time up to the singular time. In the present construction we use Craik–Criminale waves as building blocks and choose the initial velocity at each step so that the limiting solution is unforced.

Convex integration constructions of nonunique weak solutions of the Euler equations also use oscillations at successive frequency scales. Below the Onsager exponent, this method yields solutions with nonconstant kinetic energy [23, 6]. For the unforced Euler equations in the plane, Coiculescu [24,

25] uses nested vortex rings at different scales. An instability propagates from the inner rings outward, yielding distinct renormalized weak solutions with the same initial vorticity. See also Choi [26] and Hou, Qin, Song and Zhang [27]. For the three-dimensional Navier–Stokes equations, Jia and Šverák [54] gave a conditional construction of nonunique Leray–Hopf solutions based on spectral assumptions about forward self-similar flows. Nonuniqueness was shown for finite-energy weak solutions by Buckmaster and Vicol [7], and in the Leray–Hopf class with a common force by Albritton, Brué and Colombo [4]. In a recent preprint, Buckmaster, Coiculescu and Li [8] obtain nonuniqueness of unforced Leray–Hopf solutions.

1.1. **Waves and successive amplification.** We first describe the wave construction of Craik and Criminale [12]. Let $D(t)$ be a smooth matrix with $\operatorname{tr} D = 0$ and $D' + D^2$ symmetric. These conditions ensure that $U(t,x) = D(t)x$, linear in space, solves the Euler equations on $\mathbb{R}^3$, with pressure $p_D(t,x) = -x \cdot (D' + D^2)x/2$. We add to this linear background a velocity

$$w(t,x) = \mathbf{a}(t)F(k(t) \cdot x), \qquad \mathbf{a} \cdot k = 0,$$

where $k(t) \neq 0$ is the wave vector, $\mathbf{a}(t)$ is the amplitude vector, and F is a smooth odd 1-periodic profile of the scalar phase s. For example, one may take $F(s) = \sin(2\pi s)$. The vector $\mathbf{a}$ records both size and direction: the velocity amplitude is $|\mathbf{a}|\|F\|_\infty$, and $\mathbf{a}/|\mathbf{a}|$ is its unit direction when $\mathbf{a} \neq 0$. The unit wave vector is $k/|k|$. Since F has period one, w is periodic in the direction of k with period $1/|k|$, so $|k|$ is the spatial frequency of the wave.

The sum $u = Dx + w$ is an exact solution of the Euler equations if k and $\mathbf{a}$ solve

$$k' = -D^T k, \qquad \mathbf{a}' = -D\mathbf{a} + p_{\mathrm{w}} k, \qquad p_{\mathrm{w}} = \frac{2k \cdot D\mathbf{a}}{|k|^2}. \tag{1.4}$$

Its pressure is $p = p_D - p_{\mathrm{w}} P(k \cdot x)$, where $P' = F$. Indeed, $\mathbf{a} \cdot k = 0$ gives $(w \cdot \nabla)w = 0$, the first ordinary differential equation transports the phase by Dx, and the second cancels the remaining acceleration with the gradient of the wave pressure $-p_{\mathrm{w}} P(k \cdot x)$. The displayed value of p_{w} preserves $\mathbf{a} \cdot k = 0$. We call (1.4) the Craik–Criminale equations for the wave vector and the amplitude vector. Normalizing $F'(0) = 1$ gives $\nabla w(t,0) = \mathbf{a} \otimes k$. Related short-wave instabilities of planar vortices were studied by Pierrehumbert [35] and Bayly [36].

To describe amplification of a new wave, write k_p and $\mathbf{a}_p$ for the nonzero wave vector and amplitude vector of the preceding wave, and set

$$E_1 = \frac{k_p}{|k_p|}, \qquad E_2 = \frac{\mathbf{a}_p}{|\mathbf{a}_p|}, \qquad E_3 = E_1 \times E_2, \qquad \sigma = |\mathbf{a}_p|\,|k_p|.$$

E_1 is the unit wave vector and E_2 the direction vector of the preceding wave. The preceding wave contributes the velocity gradient $\mathbf{a}_p \otimes k_p = \sigma E_2 \otimes E_1$ at the origin. Here $(E_2 \otimes E_1)v = (E_1 \cdot v)E_2$: the corresponding linear velocity is $\sigma(E_1 \cdot x)E_2$, a shear in direction E_2 varying in direction E_1. The new wave therefore sees $D = B + \sigma E_2 \otimes E_1$, where B is the older background gradient. To model the wave equations at the origin, take B with just two nonzero entries in this moving frame, both constant,

$$E_1 \cdot BE_2 = \nu > 0, \qquad E_3 \cdot BE_2 = \eta > 0, \qquad \varepsilon = \eta/\nu \ll 1.$$

Start the new wave at time $t = 0$ with $k = \lambda E_3$, where $\lambda > 0$ is its initial frequency, and write $a_i = \mathbf{a} \cdot E_i$. The preceding wave evolves by (1.4) with $D = B$. Differentiating its normalized vectors gives, in this case, $E_1' = -\nu E_2$, $E_2' = \nu E_1 - \eta E_3$ and $E_3' = \eta E_2$. Combining these frame equations with (1.4) for the new wave gives, at $t = 0$,

$$\dot{a}_1 = -2\nu a_2, \qquad \dot{a}_2 = -(\sigma - \nu)a_1.$$

Freezing these coefficients gives a growing mode with rate $\sqrt{2\nu(\sigma - \nu)}$ when $\sigma > \nu$. We choose the initial direction of the amplitude along $E_2 - \sqrt{2\nu/\sigma}\,E_1$, the growing direction to leading order for $\sigma \gg \nu$.

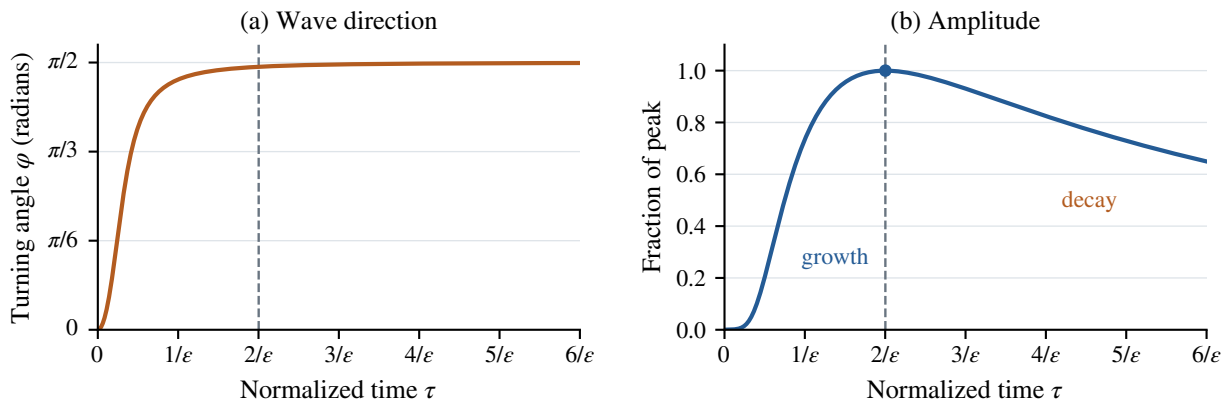


FIGURE 1. Turning and growth in the case of two nonzero entries, in the limit of a strong shear. Left: the angle φ of the wave vector from E_3 toward E_1. Right: amplitude as a fraction of its peak. The dashed line marks the amplitude peak, by which time φ is close to $\pi/2$.

The coefficients evolve as the wave vector turns. Let φ be the angle of k from E_3 toward E_1:

$$\tan\varphi = \frac{k \cdot E_1}{k \cdot E_3},$$

so $\varphi = 0$ at $t = 0$. The coupling η tilts k toward $-E_2$, and the shear carries this tilt into E_1, so φ increases; while ηt is small, $\tan\varphi$ grows like $(\sigma + \nu)\eta t^2$. In the strong-shear limit $\sigma/\nu \to \infty$, the wave pressure multiplies the driving coefficient 2ν in the equation for a_1 by $\cos^2\varphi\,(1 - \varepsilon\tan\varphi)$ (Lemma 2.3). This factor changes sign at $\tan\varphi = 1/\varepsilon$, and in the strong-shear limit the amplitude $|\mathbf{a}|$ reaches its peak after this sign change, as shown in Figure 1. For the perturbed wave with this initial direction, Proposition 2.8 gives a *gain* of at least a fixed multiple of $\exp(\varepsilon^{-1/2})$ on its prescribed interval. The gain is the ratio of the maximum of $|\mathbf{a}|$ on that interval to its initial value.

Size alone does not determine the successor wave. The new wave must supply both a large shear and a suitable frame for the next step of the iteration. In its frame (the subscript $+$ denotes the frame of the newly added wave),

$$E_{1,+} = k/|k|, \qquad E_{2,+} = \mathbf{a}/|\mathbf{a}|, \qquad E_{3,+} = E_{1,+} \times E_{2,+},$$

its shear is $\sigma_+ E_{2,+} \otimes E_{1,+}$, with $\sigma_+ = |\mathbf{a}|\,|k|$. The older gradient D, expressed in this new frame, must again have positive couplings

$$\nu_+ = E_{1,+} \cdot DE_{2,+}, \qquad \eta_+ = E_{3,+} \cdot DE_{2,+}.$$

These play the roles of ν and η in the next amplification step. We prescribe a smaller ratio $\eta_+/\nu_+ = \varepsilon_+ < \varepsilon$ to give the next wave enough amplification to keep the errors small. The turning of the wave vector supplies this orientation. In the strong-shear limit, ν_+ starts at $\varepsilon\nu$, and η_+ stays within a factor $1 + O(\varepsilon)$ of $\nu\cos\varphi$; at late times this results from a cancellation between terms much larger than $\nu\cos\varphi$. For the perturbed wave, Lemma 2.7 shows that at late times ν_+ is close to ν and η_+/ν_+ is comparable to $\cos\varphi$. Under the hypotheses of Proposition 2.8, the ratio reaches the prescribed value after the amplitude peaks in the strong-shear limit, while the new shear remains large. We start the successor when the ratio reaches that threshold. Thus the Craik–Criminale equations produce both amplification and the orientation needed to repeat the construction.

The periodic construction starts from the stationary shear flow

$$U_0(x) = \sin(2\pi x_2)(e_1 + \rho e_3),$$

where e_i are the coordinate vectors and $\rho > 0$ is small. It solves the Euler equations with zero pressure. In the initial coordinate frame its central gradient has $\nu = 2\pi$ and $\eta = 2\pi\rho$, so that $\varepsilon = \rho$. Its explicit flow supplies the first wave; subsequent waves use the successive gradients they create.

Craik–Criminale waves on $\mathbb{R}^3$ provide a local model for amplification. For the periodic construction, we localize each wave by a smooth cutoff supported in the interior of a fundamental cell in Lagrangian coordinates. We extend the localized field periodically and transport its phase and cutoff by the preceding periodic velocity, as in geometric optics [37, 38, 39, 40]. Since the cutoff vanishes near the cell boundary, the wave vector need not lie in $\mathbb{Z}^3$. Localization and the spatial variation of the background introduce errors, which the corrections below make small at each step and remove in the limit.

1.2. **Controlling corrections in an amplifying flow.** To correct a velocity in an amplifying flow, we use the freedom to choose its initial value. To avoid exciting growing modes, we prescribe the final particle positions and choose the initial velocity to reach them.

Let u, p solve the Euler equations on $[0, T]$. The flow $X(t, a)$ of u gives the position at time t of the particle that starts at a. It satisfies

$$\partial_t X(t, a) = u(t, X(t, a)), \qquad X(0, a) = a.$$

For a divergence-free velocity correction b, let X_δ be the flow of $u + \delta b$ with the same initial positions, so that

$$\partial_t X_\delta(t, a) = (u + \delta b)(t, X_\delta(t, a)), \qquad X_\delta(0, a) = a.$$

The first-order change ξ in the particle trajectories is defined by

$$X_\delta(t, a) = X(t, a) + \delta\xi(t, a) + O(\delta^2).$$

Differentiating the flow equation at $\delta = 0$ gives

$$\partial_t \xi(t, a) = \nabla u(t, X(t, a))\xi(t, a) + b(t, X(t, a)), \qquad \xi(0, a) = 0. \tag{1.5}$$

The initial condition follows from the common starting positions. Since also $X(0, a) = a$, evaluating (1.5) at $t = 0$ gives $\partial_t \xi(0, a) = b(0, a)$. This identity holds for every b and imposes no condition. We impose instead the endpoint condition $\xi(T, a) = 0$. It states that $X_\delta(T, a) = X(T, a) + O(\delta^2)$: the corrected flow carries each particle to its original final position, up to an error of order δ^2. When b solves the linearized Euler equations (1.6) below, it is determined by $b(0)$ and the source. Under the bounds $H \le \mathrm{Id}$ and $T \le 1$ below, the endpoint condition selects $b(0)$ uniquely.

For a prescribed momentum source f, the inhomogeneous problem is to find b, q satisfying the linearized Euler equations

$$\partial_t b + (u \cdot \nabla)b + (b \cdot \nabla)u + \nabla q = f, \qquad \operatorname{div} b = 0. \tag{1.6}$$

Here q is the pressure variation, and the displacement induced by b must satisfy $\xi(0) = \xi(T) = 0$. Differentiating (1.5) in time and using (1.6) gives

$$\begin{aligned} \xi_{tt} + H\xi + (\nabla q) \circ X = f \circ X, \qquad H = (D_x^2 p) \circ X, \\ \operatorname{div}_x(\xi \circ X^{-1}) = 0, \qquad \xi(0) = \xi(T) = 0. \end{aligned} \tag{1.7}$$

The homogeneous equation is the Jacobi equation along the geodesic associated with this solution in Arnold's formulation [5]. The constraint $\operatorname{div}_x(\xi \circ X^{-1}) = 0$ expresses volume preservation to first order.

For a solution of (1.7), testing with $-\xi$ and integrating by parts gives the identity

$$\underbrace{\int_0^T \int_{\mathbb{T}^3} \left(|\xi_t|^2 - \xi \cdot H\xi\right) \, \mathrm{d}a \, \mathrm{d}t}_{\mathcal{Q}(\xi)} = -\int_0^T \int_{\mathbb{T}^3} (f \circ X) \cdot \xi \, \mathrm{d}a \, \mathrm{d}t. \tag{1.8}$$

Volume preservation and the divergence constraint remove the pressure variation. The quadratic form $\mathcal{Q}$ is the fixed-endpoint second variation of the action. If $H \leq \mathrm{Id}$ and $T \leq 1$, then $|\xi(t,a)|^2 \leq (T-t)\int_t^T |\xi_\tau(\tau,a)|^2 \, \mathrm{d}\tau$ gives

$$\mathcal{Q}(\xi) \geq \left(1 - \frac{T^2}{2}\right) \|\xi_t\|_{L^2_{t,a}}^2 \geq \frac{1}{2} \|\xi_t\|_{L^2_{t,a}}^2. \tag{1.9}$$

The term $-\xi \cdot H\xi$ lowers the integrand where $\xi \cdot H\xi > 0$ and raises it where $\xi \cdot H\xi < 0$. Thus an upper bound for H gives the lower bound in (1.9). For $f = 0$, (1.8) gives $\mathcal{Q}(\xi) = 0$. Then (1.9) forces $\xi_t = 0$, and $\xi(0) = 0$ gives $\xi = 0$. Equation (1.5) gives $b = 0$, proving that the linearized endpoint map $b(0) \mapsto \xi(T)$ has trivial kernel. Upper bounds on the pressure Hessian also enter Brenier's fixed-endpoint action-minimization criterion [32, 33].

For an approximate parent U with residual R_U, Subsection 5.1 uses $H^{\#} = (D_x^2 p_U - \nabla R_U) \circ X$ in place of H, under the same bounds $H^{\#} \leq \mathrm{Id}$ and $T \leq 1$. Subsection 4.3 uses the resulting coercivity to construct a bounded inverse for the forced Jacobi problem, and Appendix A establishes the quantitative Hölder bounds of Theorem 4.2. The inverse exists on every interval $[0,T]$ with $0 < T \leq 1$ on which $H^{\#} \leq \mathrm{Id}$, with a bound that depends on the parent but requires no further smallness of T. Each correction therefore solves a linear two-point boundary value problem along the flow of the parent, and no short-time argument is used.

In the affine wave of Subsection 1.1, the Hessian of $-p_{\mathrm{w}} P(k \cdot x)$ is $-p_{\mathrm{w}} F'(k \cdot x) k \otimes k$. We seek an odd periodic profile F_h satisfying

$$F_h'(0) = 1, \qquad -Ch \leq F_h'(s) \leq 1. \tag{1.10}$$

The first condition preserves the wave gradient $\mathbf{a} \otimes k$ at the origin, and hence the amplification described in Subsection 1.1. The lower bound on F_h' controls the largest eigenvalue of $D_x^2[-p_{\mathrm{w}} P_h(k \cdot x)]$, where $P_h' = F_h$. On an interval with $p_{\mathrm{w}} > 0$, its positive part satisfies

$$p_{\mathrm{w}} |k|^2 \left(-F_h'(k \cdot x)\right)_+ \leq Ch \, p_{\mathrm{w}} |k|^2, \tag{1.11}$$

where $(r)_+ = \max\{r, 0\}$. For fixed wave coefficients, taking h small makes this positive part small at every point. In the construction h is a negative power of the frequency of the wave. As h shrinks, the wave produces thin layers of strong shear while its velocity amplitude is $O(h|\mathbf{a}|)$. Its gradient at the origin remains $\mathbf{a} \otimes k$, preserving the local shear that amplifies the next wave. The pressure Hessian added at each step is bounded above by a small multiple of the identity. Section 7 sums these bounds and controls $\|\nabla R_U\|_\infty$ to obtain $H^{\#} \leq \mathrm{Id}$ as a quadratic form.

For $F(s) = \sin(2\pi s)$, the ratio $F'(1/2)/F'(0)$ is -1. The sine profile with slope one at the origin therefore violates the lower bound in (1.10) for small h. Subsection 3.2 constructs F_h with a narrow positive bump in F_h' and a shallow negative part, as shown in Figure 2.

1.3. **Correcting a localized wave.** We smoothly truncate the velocity of each step in frequency at the end of the step, to simplify the bookkeeping of derivative estimates, as in the mollification step of convex integration [6, Section 2.4]. This introduces a small error, carried to the next step; Subsection 5.7 gives the estimates. Write U for the truncated velocity of the preceding step and R_U for its residual in the Euler equations.

We add the localized wave described in Subsection 1.1, using F_h and a divergence correction. Let X be the flow of U from time zero, and fix a nonzero vector k_*. The phase θ of the localized wave is defined by

$$\theta(t, X(t,a)) = k_* \cdot a$$

for Lagrangian coordinates a in the fundamental cell. It is the plane phase $k_* \cdot x$ at time zero, transported by U. Its gradient $\kappa = \nabla_x \theta$ is the wave vector of the localized wave; for a linear flow, $\theta = \kappa \cdot x$, as in Subsection 1.1. For a wave after the first, k_* is chosen so that $\kappa = \lambda E_3$ at the origin at its activation time, as for the new wave of Subsection 1.1. The leading velocity is $A_1(t, x, \theta(t, x))$. Writing the profile as $A_1(t, x, s)$ with s independent lets us distinguish derivatives of the coefficients from the large term $\kappa \partial_s$ that arises when differentiating through the phase. A wave after the first is multiplied by a cutoff $\widetilde{\chi}(t)$ in time that vanishes before its amplification interval; for the first wave $\widetilde{\chi} = 1$. The derivative of the cutoff leaves an error of the size of the wave at the start of that interval, also carried to the next step.

The velocity added to U is a finite expansion

$$\sum_{p=1}^{P} Z_p + C_{P+1}, \qquad Z_p = V_p + C_p + B_p,$$

where $V_p(t, x, s)$ is a profile perpendicular to κ with zero phase average, C_{p+1} is the divergence correction of V_p, and $B_p(t, x)$ is a mean, independent of the phase; the profiles are evaluated at $s = \theta$. The leading term is the wave, $Z_1 = V_1 = \widetilde{\chi} A_1$, with $C_1 = 0$ and $B_1 = 0$. We call p the *level* of Z_p and of each of V_p, C_p and B_p: up to a fixed power of a number that bounds the parent, the bound for Z_p is that of the wave times $\lambda^{-2(p-1)/3}$, where λ is the frequency of the wave. The corrections are computed one level at a time up to the depth P, which is fixed in Section 7. The error left by the wave and its divergence correction C_2 is a profile $H_2(t, x, s)$ of level two. We split H_2 into its phase average $\langle H_2 \rangle_s$ and its oscillatory part $H_2 - \langle H_2 \rangle_s$, as in the expansions of Cheverry [34, Section 4.1].

The mean correction B_2 cancels the phase average of H_2 together with the parent residual. With $D_t = \partial_t + U \cdot \nabla_x$, the linearized Euler equation (1.6) gives $D_t B_2 + (\nabla U) B_2 + \nabla q = -\langle H_2 \rangle_s - R_U$ with $\operatorname{div} B_2 = 0$. The Jacobi inverse selects $B_2(0)$ so that the correction leaves the truncated parent's final particle positions fixed to first order.

With B_2 determined, the oscillatory correction V_2 and its pressure π_2 solve

$$\begin{aligned} D_t V_2 + (\nabla U) V_2 + \kappa \partial_s \pi_2 &= -(H_2 - \langle H_2 \rangle_s) - (B_2 \cdot \kappa) \partial_s (\widetilde{\chi} A_1), \\ \kappa \cdot V_2 = 0, \qquad \langle V_2 \rangle_s &= 0. \end{aligned} \tag{1.12}$$

The mean B_2 transports the leading wave's fast phase. This term has zero phase average and, like the oscillatory part $H_2 - \langle H_2 \rangle_s$, belongs to level two, so we solve for the mean B_2 before the oscillatory correction V_2. At fixed phase along each trajectory, (1.12) is a forced Craik–Criminale amplitude equation. Every wave uses zero initial data for this correction. The curl construction of

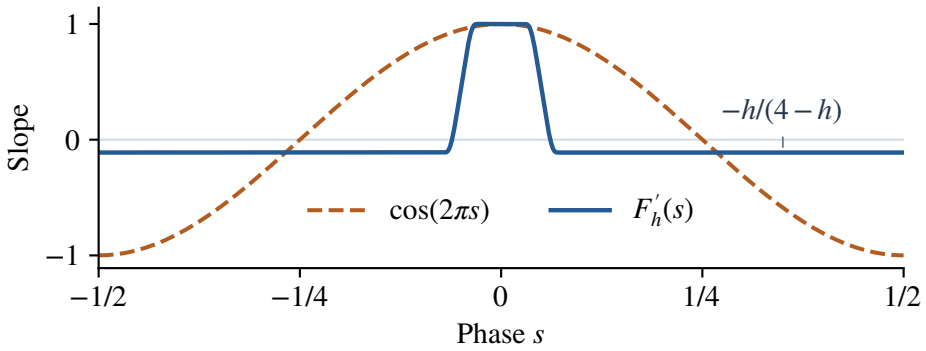


FIGURE 2. The sine slope divided by its value at the origin (dashed) and the slope F_h' of the modified profile (solid). Both curves equal one at the origin.

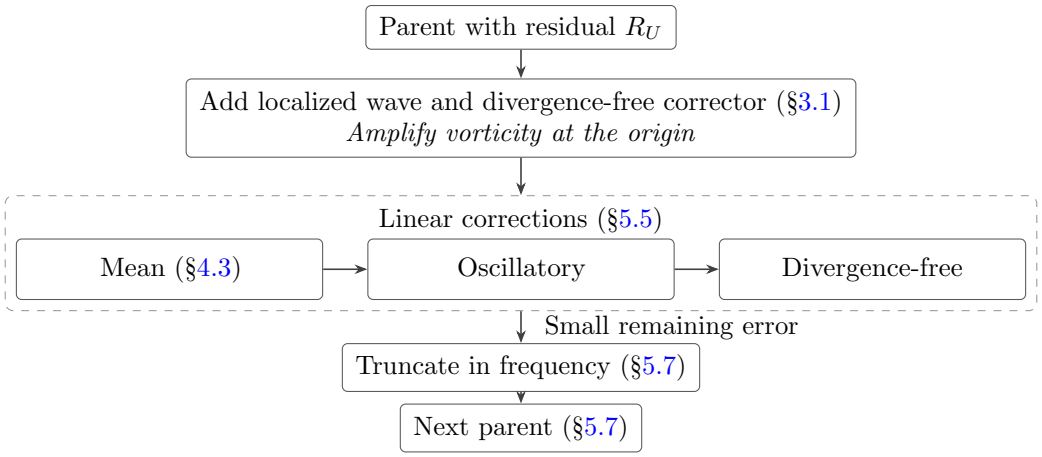


FIGURE 3. One step of the construction.

Subsection 3.1 gives the divergence correction C_3 of V_2, so that $V_2 + C_3$ is exactly divergence free. The term C_3 belongs to level three, where we solve for the mean and oscillatory corrections again.

We stop the expansion at the depth P, keeping the divergence correction C_{P+1} of V_P. The remainder then has a small factor $\lambda^{-2P/3}$, once λ is larger than a fixed power of a number that bounds the parent (Theorem 5.6). The truncated parent, localized wave and linear corrections give an approximate solution V of the Euler equations. We truncate V in frequency. Three choices make the three errors of the step smaller than a fixed negative power of the frequency of the next wave. The depth P controls the remainder of the expansion. The frequency cutoff controls the truncation error. The ratio ε of Subsection 1.1 sets the gain, which controls the error of the cutoff in time. This cutoff acts while the wave's amplitude is at most a fixed multiple of $\exp(-\varepsilon^{-1/2})$ times its peak. These three errors make up the residual of the truncated velocity, which is the parent of the next step (Proposition 5.11). The pressure and residual-gradient estimates give coercivity for the mean equation of the next step. The leading wave and oscillatory corrections stay in the transported cutoff region. The mean corrections and the truncation difference may extend over the torus. Figure 3 summarizes the step.

1.4. **Comparison with blow-up constructions with forcing.** The construction of blow-up for the forced Boussinesq equations by Alpöge and Buckmaster [1, Section 1.2] also uses coupled wave amplitudes to amplify a gradient and prepare the next wave. There the temperature gradient grows together with a vorticity amplitude. The accompanying shear would obstruct the next growth stage, so an external force rotates the temperature gradient and wave direction together. This preserves the coupling that drives the temperature amplitude while changing the horizontal component of the wave vector, and hence the sign of vorticity production. A scalar shooting parameter selects a rotation pulse that returns the vorticity amplitude and that horizontal component to zero, retaining the temperature gain. Keeping the wave vector vertical then holds both amplitudes fixed while the next wave grows.

The construction of blow-up for the forced Euler equations by Alpöge and Buckmaster [2, Section 1.2] builds on this mechanism for axisymmetric flow with swirl. Circulation replaces temperature and couples to azimuthal vorticity. The angular control returns the principal azimuthal-vorticity amplitude at the material center to zero while retaining the circulation-gradient gain. The radial dependence of the elliptic operator and of this coupling requires amplitudes that vary across the transported support, with transport by the preceding velocity, including its corrections.

In the present unforced construction, the amplified shear itself drives the successor. Its orientation is obtained from the Craik–Criminale wave evolution; the pressure coefficient is determined by

incompressibility and is not a rotation control. The next wave is activated at the start of its amplification interval, with its amplitude in the growing direction (Subsection 6.2).

Localization presents a further difference. In the construction for the Boussinesq equations the profiles are linear near zero, so each layer's velocity and temperature are exactly linear on a region containing the later layers. The streamfunction corrections vanish there, preserving this region. In both forced constructions, the remaining errors are absorbed by a force smooth through the singular time. To solve the unforced Euler equations, we must remove those errors. The pressure and velocity corrections need not preserve a neighborhood on which the velocity is linear; the next wave uses the central gradient and estimates on its spatial variation.

The organization of this paper is as follows: Section 2 proves amplification and its stability, Sections 3–5 localize the waves and construct the linear corrections, Sections 6 and 7 start and iterate the construction, and Section 8 passes to the limit.

The public repository https://github.com/mcoiculescu/EulerBlowupPublic contains a Lean formalization of the proof and the latest version of this manuscript.

Conventions. A time supremum is indicated by C_t or by the time interval. The Leray projection is $\mathbb{P} = I - \nabla\Delta^{-1}\operatorname{div}$, where Δ^{-1} acts on zero-mean scalars. The periodic velocities constructed below are odd and hence have zero spatial mean. Pressure is normalized to have zero spatial mean. Spatial differences use the periodic metric, and flows are lifted so that their displacements are periodic. We use $(\nabla U)_{ij} = \partial_j U_i$, and for a matrix field A, $(\operatorname{div} A)_i = \sum_j \partial_j A_{ij}$. A matrix acts on vectors by $(Av)_i = \sum_j A_{ij} v_j$. For vectors a, b we write $a \otimes b = ab^T$, so that $(a \otimes b)_{ij} = a_i b_j$ and $(a \otimes b)v = (b \cdot v)a$. For an orthonormal basis (E_1, E_2, E_3), the entries of A in this basis are $A_{ij} = E_i \cdot AE_j$; then $AE_j = \sum_i A_{ij} E_i$, $A^T E_i = \sum_j A_{ij} E_j$ and $Av = \sum_{i,j} A_{ij}(E_j \cdot v)E_i$.

1.5. **Choice of parameters.** The frequencies and orders are chosen so that the truncation and correction errors are summable. The gains also make each new wave small at its activation time. We fix the scales here; Appendix C proves Lemma 1.3.

Fix the smooth profiles used in the construction. The integer $b \geq 1$ is the fixed integer specified in Subsection 5.6. For integers $s \geq 1$ put $K(s) = 320b(s+1)$. The exponent K separates successive frequencies.

The estimates of Sections 3, 5, 6 and 7 and of Lemma B.1 contain constants that depend only on an order of differentiation, the depth of the expansion and the fixed profiles. We call them the fixed-order constants. Choose a nondecreasing function $\mathcal{B}$ on the positive integers such that $\mathcal{B}(s)$ bounds the fixed-order constants at all orders and depths at most $7K(s)$, and

$$\log \mathcal{B}(s) \geq 4K(s+1)^3 \qquad (s \geq 1). \tag{1.13}$$

Let $N \geq \mathcal{B}(1)$ be the initial frequency parameter. The integer q_j specifies the spatial derivative order through which step j makes only a small change to the velocity on earlier time intervals. We define the frequencies and q_j together, starting from $q_1 = 1$. Put

$$\begin{gathered} K_j = K(q_j), \qquad \lambda_0 = N, \qquad \lambda_j = \lambda_{j-1}^{K_j} \quad (j \geq 1), \qquad \Lambda_j = \lambda_j^5 \quad (j \geq 0), \\ q_{j+1} = \max\{1 \leq s \leq q_j + 1 : \ \mathcal{B}(s) \leq \lambda_{j-1}\} \quad (j \geq 1), \\ L_j = 4 \log \lambda_{j+1}, \qquad \varepsilon_{j-1} = L_j^{-2} \quad (j \geq 2). \end{gathered} \tag{1.14}$$

Here λ_j, $j \geq 1$, is the frequency of wave j, and L_j is the prescribed logarithmic gain. For $j \geq 1$, Λ_j is the Fourier cutoff applied to the velocity constructed at step j, and Λ_0 bounds the frequencies of the shear from which the construction starts.

The orders q_j must tend to infinity for the limit to be smooth; however fast $\mathcal{B}$ grows, it only slows their growth. They must also grow slowly: the amplification estimates of Section 7 lose fixed powers of the logarithmic gain L_{j+1}, which grows with the orders through the exponents K_j, and these losses must stay below small powers of λ_{j-1}. The lower bound (1.13) is imposed for this purpose.

Lemma 1.3. *The integers q_j satisfy $q_j \le q_{j+1} \le q_j + 1$ and tend to infinity, and*

$$L_{j+1} \le (\log \lambda_{j-1})^2 \qquad (j \ge 1). \tag{1.15}$$

1.6. **Provenance.** Before OpenAI released its paper on the unforced Euler equations, the first and second authors had completed a second proof of blow-up for the Euler equations with smooth forcing. Unlike their public proof [2], which uses axisymmetric flow with swirl, this unreleased proof uses the same Craik–Criminale amplification mechanism as OpenAI's paper, including the same reduced equations for the amplitude of a wave, the system (2.5) below. In our unpublished work on the hypodissipative Navier–Stokes equations, much of the argument for removing the force was developed, with human prompts guiding that work. This is consistent with the second author's statement of September 7 that the hypodissipative work pointed toward a blow-up construction for the unforced Euler equations [30].

The second author's account of his communications with OpenAI about these results is given in [30].

1.7. **AI statement.** We used Claude and Codex throughout this work. The first proof without force was written by Anthropic models from the drafts on the Euler equations with forcing and on the hypodissipative Navier–Stokes equations described in Subsection 1.6. Like the first writeups of our proofs for the Boussinesq and Euler equations, it was long and hard to read. We then rewrote it with Claude and Codex under our direction, iterating with the models on the structure of the argument, isolating the key ideas, and simplifying it over successive revisions. In the course of this we replaced parts of the argument by ideas of our own, which the models implemented. Together with our own editing, this led to the present text.

2. The wave and its amplification

2.1. **Exact waves over a linear background.** We first work on $\mathbb{R}^3$, where linear velocity fields give local models for the periodic construction. Let $D(t)$ be smooth and trace free, with $D' + D^2$ symmetric. Then Dx solves the Euler equations with pressure $-x \cdot (D' + D^2)x/2$. Choose a smooth odd 1-periodic profile F with $F'(0) = 1$, and let $P' = F$, $\langle P \rangle_s = 0$. Here $\langle H \rangle_s = \int_0^1 H(s)\,\mathrm{d}s$ is the phase average.

Lemma 2.1. *Let the wave vector $k(t) \neq 0$ and the amplitude vector $\mathbf{a}(t)$, functions of time alone, satisfy*

$$k' = -D^T k, \qquad \mathbf{a}' = -D\mathbf{a} + p_{\mathrm{w}} k, \qquad p_{\mathrm{w}} = \frac{2k \cdot D\mathbf{a}}{|k|^2}, \qquad k \cdot \mathbf{a} = 0. \tag{2.1}$$

Then $Dx + \mathbf{a}F(k \cdot x)$ is divergence free and solves the Euler equations on $\mathbb{R}^3$ with pressure $-x \cdot (D' + D^2)x/2 - p_{\mathrm{w}} P(k \cdot x)$. If $k(t_0) \neq 0$, $\mathbf{a}(t_0) \neq 0$ and $k(t_0) \cdot \mathbf{a}(t_0) = 0$, then both vectors remain nonzero and perpendicular for every $t \ge t_0$ in the interval of existence of D.

Proof. Write $w = \mathbf{a}F(k \cdot x)$ and $u = Dx + w$. Since $k' = -D^T k$,

$$(\partial_t + Dx \cdot \nabla)(k \cdot x) = (k' + D^T k) \cdot x = 0, \qquad (\partial_t + Dx \cdot \nabla) w = \mathbf{a}' F.$$

Also $(w \cdot \nabla) Dx = D\mathbf{a}F$, and perpendicularity cancels the self-advection,

$$(w \cdot \nabla) w = (k \cdot \mathbf{a}) F F' \mathbf{a} = 0.$$

Consequently

$$\partial_t u + (u \cdot \nabla) u = (D' + D^2)x + (\mathbf{a}' + D\mathbf{a})F.$$

The first term is minus the gradient of the background pressure, and the second equals $p_{\mathrm{w}} k F$, which is minus the gradient of the wave pressure $-p_{\mathrm{w}} P(k \cdot x)$. Moreover $\operatorname{div} u = \operatorname{tr} D + (k \cdot \mathbf{a}) F' = 0$. The choice of p_{w} gives $(k \cdot \mathbf{a})' = -2k \cdot D\mathbf{a} + p_{\mathrm{w}} |k|^2 = 0$. For a given D, the equation for k is linear and has an invertible fundamental matrix. Hence $k(t_0) \neq 0$ implies $k(t) \neq 0$. With this k, the equation

$\mathbf{a}' = -(\mathrm{Id} - 2\frac{k}{|k|}\frac{k^T}{|k|})D\mathbf{a}$ is also linear with an invertible fundamental matrix, so $\mathbf{a}(t_0) \neq 0$ implies $\mathbf{a}(t) \neq 0$. □

For a single profile, this is the inviscid wave construction of Craik and Criminale [12].

2.1.1. *The preceding frame.* A wave evolving over a background matrix B becomes part of the background for the next wave. At the origin, its contribution to the velocity gradient is $\mathbf{a} \otimes k$, and its unit wave vector and velocity direction provide a moving orthonormal frame. We express the older gradient B in this frame to identify the couplings that drive the next wave. The following calculation uses only $\operatorname{tr} B = 0$; it requires no symmetry of $B' + B^2$.

Lemma 2.2. *Let $B(t)$ be a smooth trace-free matrix, and let $k \neq 0$ and $\mathbf{a} \neq 0$ satisfy (2.1) with D replaced by B. Put $E_1 = \frac{k}{|k|}$, $E_2 = \frac{\mathbf{a}}{|\mathbf{a}|}$ and $E_3 = E_1 \times E_2$, write $B_{ij} = E_i \cdot BE_j$ $(i, j = 1, 2, 3)$ for the entries of B in this orthonormal frame, and let $\sigma = |k|\,|\mathbf{a}|$, $\nu = B_{12}$ and $\eta = B_{32}$. Then*

$$E_1' = -\nu E_2 - B_{13}E_3, \quad E_2' = \nu E_1 - \eta E_3, \quad E_3' = B_{13}E_1 + \eta E_2, \qquad (\log\sigma)' = B_{33}. \tag{2.2}$$

Proof. For an arbitrary nonzero vector $v(t)$ depending on time, put $\widehat{v} = v/|v|$. In this proof primes denote derivatives in t. Then

$$\widehat{v}' = (\mathrm{Id} - \widehat{v}\widehat{v}^T)\frac{v'}{|v|}, \qquad (\log|v|)' = \widehat{v} \cdot \frac{v'}{|v|}.$$

Since $B^T E_1 = \sum_j B_{1j}E_j$ and $k' = -B^T k$, this gives

$$E_1' = -(\mathrm{Id} - E_1E_1^T)B^T E_1 = -B_{12}E_2 - B_{13}E_3, \qquad (\log|k|)' = -B_{11}.$$

Similarly $BE_2 = \sum_i B_{i2}E_i$ and $p_{\mathrm{w}}|k| = 2|\mathbf{a}|B_{12}$, so

$$E_2' = (\mathrm{Id} - E_2E_2^T)(-BE_2 + 2B_{12}E_1) = B_{12}E_1 - B_{32}E_3, \qquad (\log|\mathbf{a}|)' = -B_{22}.$$

Using $E_3 \times E_2 = -E_1$ and $E_1 \times E_3 = -E_2$ gives

$$E_3' = E_1' \times E_2 + E_1 \times E_2' = B_{13}E_1 + B_{32}E_2.$$

Finally, $(\log\sigma)' = -B_{11} - B_{22} = B_{33}$ because $\operatorname{tr} B = 0$. □

2.2. **Amplification on a prescribed interval.** Of the two entries $\nu = E_1 \cdot BE_2$ and $\eta = E_3 \cdot BE_2$ of the older gradient introduced in Lemma 2.2, the coupling ν helps the preceding shear amplify a new wave, while η changes its direction and eventually ends the growth. A small ratio η/ν therefore allows a large gain before decay. In the model case of Subsection 1.1 and the limit of an infinitely strong preceding shear, the amplitude of a new wave solves a linear system of two equations with explicit coefficients (Lemma 2.3). Up to errors measured by one number δ, the amplitude of a general new wave solves the same system (Lemma 2.4). Its solution with growing initial data peaks after a large gain (Lemma 2.5), and the wave follows this solution with relative errors that grow only polynomially in the length of the interval (Lemma 2.6). The couplings of the new wave are estimated in Lemma 2.7, and Proposition 2.8 locates the ratio prescribed for its successor after the amplitude peaks in the strong-shear limit.

2.2.1. *The new wave in the preceding frame.* Let $B(t)$ be a smooth trace-free matrix, and let the preceding wave $(k_p, \mathbf{a}_p)$, with $k_p \neq 0$ and $\mathbf{a}_p \neq 0$, satisfy (2.1) with D replaced by B. Let (E_1, E_2, E_3) be its frame, $B_{ij} = E_i \cdot BE_j$ the entries of B in this frame, and $\sigma_p = |k_p|\,|\mathbf{a}_p|$ its strength, as in Lemma 2.2. Thus (2.2) holds with the functions B_{12}, B_{32} and σ_p in the places of ν, η and σ. The new wave evolves over

$$M = B + \mathbf{a}_p \otimes k_p + E_c = B + \sigma_p E_2 \otimes E_1 + E_c,$$

the sum of the older gradient, the gradient of the preceding wave at the origin, and a continuous matrix $E_c(t)$ of errors. Its wave vector k solves the first equation of (2.1) with $D = M$ and equals

λE_3, with $\lambda > 0$, at the activation time t_0. For this k, an *amplitude vector* is a solution $\mathbf{a}$ of the second equation in (2.1) with $D = M$ satisfying $\mathbf{a}(t) \cdot k(t) = 0$. Only these ordinary differential equations enter this subsection: the matrices B and E_c are prescribed, and no equation is assumed for them. Subsection 6.2 applies the results below with M the velocity gradient of the parent at the origin, which is a fixed point of the flow.

Freeze the strength and the two couplings at the activation time,

$$\sigma = \sigma_p(t_0), \qquad \nu = B_{12}(t_0) > 0, \qquad \eta = B_{32}(t_0) > 0,$$

and put

$$r = \sqrt{2\sigma\nu}, \qquad S = \frac{\sigma}{r} = \sqrt{\frac{\sigma}{2\nu}}, \qquad \varepsilon = \frac{\eta}{\nu}, \qquad \tau = r(t - t_0).$$

Since $\nu/\sigma = 1/(2S^2)$, the growth rate $\sqrt{2\nu(\sigma - \nu)}$ found in Subsection 1.1 equals $r(1 - 1/(2S^2))^{1/2}$; thus r is this rate up to a factor close to one when S is large. The variable τ measures time in units of the growth time r^{-1}, and S is the ratio of the preceding shear rate to the growth rate. We work on the interval $0 \le \tau \le T$, or $t_0 \le t \le t_0 + T/r$, with $T \ge 2$. In (2.1) and (2.2) primes denote derivatives in t. From now on in this subsection they denote derivatives in τ, so that $d/d\tau = r^{-1} d/dt$. All suprema are over $0 \le \tau \le T$, and $b_{ij} = B_{ij}/r$.

Define the scaled components $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of the wave vector and an amplitude vector by

$$k = \lambda \left(\zeta_1 E_1 + \frac{\zeta_2}{S} E_2 + \zeta_3 E_3 \right), \qquad \mathbf{a} = \frac{\alpha_1}{S} E_1 + \alpha_2 E_2 + \frac{\alpha_3}{S} E_3. \tag{2.3}$$

Then $\zeta(0) = (0, 0, 1)$ and

$$|k|^2 = \lambda^2 m, \qquad m = \zeta_1^2 + \zeta_2^2/S^2 + \zeta_3^2.$$

Transversality is $\zeta \cdot \alpha = 0$, so where $\zeta_3 \neq 0$,

$$\alpha_3 = -\frac{\zeta_1 \alpha_1 + \zeta_2 \alpha_2}{\zeta_3}.$$

The two components α_1, α_2 therefore determine the amplitude vector. The factor S^{-1} in the E_2 coordinate of k makes the contribution of B to the coefficient of ζ_3 in the equation for ζ_2 equal to $-B_{32}/\nu$, which equals $-\varepsilon$ at $\tau = 0$. In the growing mode of Subsection 1.1, the components of the amplitude along E_1 and E_2 have opposite signs, and the first is $S^{-1}(1 - \nu/\sigma)^{-1/2}$ times the second in absolute value. With the scaling in (2.3), this mode is $\alpha_1 = -\alpha_2$ to leading order. In Subsection 2.2.2 the components (α_1, α_2) are divided further by the amplitude in the strong-shear limit, which gives the normalized coordinates (2.15).

In the model case of Subsection 1.1, the entries of B in the frame are the constants $B_{12} = \nu$ and $B_{32} = \eta$, all other entries vanish, and $E_c = 0$. Put

$$\tan\varphi = \tfrac{1}{2}\varepsilon\tau^2, \qquad 0 \le \varphi < \tfrac{\pi}{2}, \qquad \zeta_0 = (\tan\varphi, -\varepsilon\tau, 1). \tag{2.4}$$

The next lemma derives the equations for the scaled components in the model case and their limit as the preceding shear becomes infinitely strong. In the limit the wave vector is $\lambda(\tan\varphi E_1 + E_3)$, so φ is its angle from E_3 toward E_1, as in Subsection 1.1.

Lemma 2.3. *Suppose that on $0 \le \tau \le T$ the entries of B in the frame are $B_{12} = \nu$, $B_{32} = \eta$ and $B_{ij} = 0$ otherwise, and that $E_c = 0$. Then $\sigma_p = \sigma$, and the scaled components of the wave vector satisfy*

$$\zeta_1' = -\Big(1 + \frac{1}{2S^2}\Big)\zeta_2, \qquad \zeta_2' = -\varepsilon\zeta_3, \qquad \zeta_3' = \frac{\varepsilon}{2S^2}\zeta_2.$$

Where $\zeta_3 \neq 0$, the components α_1, α_2 of every amplitude vector satisfy

$$\alpha_1' = -\alpha_2 + \zeta_1 \Pi,$$
$$\alpha_2' = -\Big(1 - \frac{1}{2S^2}\Big)\alpha_1 + \frac{\varepsilon}{2S^2}\,\frac{\zeta_1\alpha_1 + \zeta_2\alpha_2}{\zeta_3} + \frac{\zeta_2}{S^2}\,\Pi, \qquad \Pi = \frac{2\zeta_2\alpha_1 + (\zeta_1 + \varepsilon\zeta_3)\alpha_2}{m}.$$

As $S \to \infty$ with ε and T fixed, $\zeta \to \zeta_0$ uniformly on $[0,T]$, and the coefficients of this system for (α_1, α_2) converge uniformly on $[0,T]$ to those of

$$\alpha_1' = -\cos^2\varphi(1 - \varepsilon\tan\varphi)\alpha_2 - \varepsilon\tau\sin 2\varphi\,\alpha_1, \qquad \alpha_2' = -\alpha_1. \tag{2.5}$$

Proof. In this proof write $k_i = k \cdot E_i$ and $a_i = \mathbf{a} \cdot E_i$, and let a dot denote the derivative in t. The frame equations (2.2) read $\dot E_1 = -\nu E_2$, $\dot E_2 = \nu E_1 - \eta E_3$ and $\dot E_3 = \eta E_2$, and their last equation gives $d(\log\sigma_p)/dt = B_{33} = 0$, so $\sigma_p = \sigma$. Thus $M = \nu E_1 \otimes E_2 + \eta E_3 \otimes E_2 + \sigma E_2 \otimes E_1$, and $ME_1 = \sigma E_2$, $ME_2 = \nu E_1 + \eta E_3$, $ME_3 = 0$. Since $\dot k = -M^T k$, we have $\dot k_i = -k \cdot ME_i + k \cdot \dot E_i$, hence

$$\dot k_1 = -(\sigma + \nu)k_2, \qquad \dot k_2 = -2\eta k_3, \qquad \dot k_3 = \eta k_2.$$

Since $\dot{\mathbf{a}} = -M\mathbf{a} + p_{\mathrm{w}} k$ and $M\mathbf{a} = \nu a_2 E_1 + \sigma a_1 E_2 + \eta a_2 E_3$, also $\dot a_i = -E_i \cdot M\mathbf{a} + p_{\mathrm{w}} k_i + \mathbf{a} \cdot \dot E_i$, hence

$$\dot a_1 = -2\nu a_2 + p_{\mathrm{w}} k_1, \qquad \dot a_2 = -(\sigma - \nu)a_1 - \eta a_3 + p_{\mathrm{w}} k_2, \qquad p_{\mathrm{w}} = \frac{2(\nu k_1 a_2 + \sigma k_2 a_1 + \eta k_3 a_2)}{|k|^2}.$$

Substitute (2.3), divide by r to pass to the time τ, and use

$$rS = \sigma, \qquad \frac{\nu}{\sigma} = \frac{1}{2S^2}, \qquad \frac{\nu S}{r} = \frac{1}{2}, \qquad \frac{\eta S}{r} = \frac{\varepsilon}{2}, \qquad \frac{\eta}{\sigma} = \frac{\varepsilon}{2S^2}.$$

The equations for k_i give those for ζ, and $\lambda S p_{\mathrm{w}}/r = \Pi$. The equations for a_1 and a_2 then give those for $\alpha_1 = Sa_1$ and $\alpha_2 = a_2$, after substituting $\alpha_3 = -(\zeta_1\alpha_1 + \zeta_2\alpha_2)/\zeta_3$.

As $S \to \infty$, the coefficients of the system for ζ converge to those of $\zeta_1' = -\zeta_2$, $\zeta_2' = -\varepsilon\zeta_3$, $\zeta_3' = 0$, whose solution with $\zeta(0) = (0,0,1)$ is ζ_0. Both systems are linear with bounded coefficients on $[0,T]$, so $\zeta \to \zeta_0$ uniformly there by Gronwall's inequality. In particular $\zeta_3 \to 1$ and $m \to 1 + \tan^2\varphi = \cos^{-2}\varphi$. In the equation for α_1 the coefficients of α_1 and α_2 therefore converge to

$$\frac{2\zeta_1\zeta_2}{m} \to -2\varepsilon\tau\tan\varphi\cos^2\varphi = -\varepsilon\tau\sin 2\varphi, \qquad -1 + \frac{\zeta_1(\zeta_1 + \varepsilon\zeta_3)}{m} \to -\cos^2\varphi(1 - \varepsilon\tan\varphi),$$

since $(\tan\varphi + \varepsilon)\tan\varphi\cos^2\varphi = \sin^2\varphi + \varepsilon\sin\varphi\cos\varphi$. In the equation for α_2 every term except $-\alpha_1$ has a factor S^{-2} and a bounded coefficient, so the two coefficients converge to -1 and 0. □

We call (2.5) the *reference system*. The vector ζ_0 solves the limiting system $\zeta_1' = -\zeta_2$, $\zeta_2' = -\varepsilon\zeta_3$, $\zeta_3' = 0$ for the wave vector.

The errors of the comparison are measured by

$$\delta = \Big\|\frac{B_{12}}{\nu} - 1\Big\|_\infty + \Big\|\frac{B_{32}}{\eta} - 1\Big\|_\infty + \frac{\|B\|_\infty}{r} + \frac{\|E_c\|_\infty}{\nu}. \tag{2.6}$$

The first two terms are the relative variations of the two couplings on the interval. The third compares the older gradient with the growth rate; it is at least $\nu/r = 1/(2S)$, as shown below, so it is small only if the preceding shear is strong. The last compares the error E_c with the coupling ν rather than with r. The scaled components multiply the E_2 component of k and the E_1 and E_3 components of $\mathbf{a}$ by S, so in the time τ an entry of E_c enters their equations with a factor up to $S/r = 1/(2\nu)$; see the proof of Lemma 2.4.

Fix an absolute constant C_0, sufficiently large for the finitely many estimates of this subsection, and assume

$$C_0\varepsilon \le 1, \qquad C_0 T^{C_0}\delta \le \varepsilon. \tag{2.7}$$

All other constants and exponents C, c in this subsection are absolute and do not depend on C_0; they may change from one estimate to the next. The errors below have the form $CT^C\delta$, and by (2.7) such an error is at most ε times any prescribed negative power of T once C_0 is large.

Condition (2.7) forces the preceding shear to be strong. Indeed, $|B| \geq |E_1 \cdot BE_2|$, and at $\tau = 0$ this entry is $B_{12}(t_0) = \nu$. Since $\nu/r = \sqrt{\nu/(2\sigma)} = 1/(2S)$, the third term of (2.6) gives $\delta \geq \nu/r = 1/(2S)$. On the other hand, (2.7) gives $\delta \leq \varepsilon C_0^{-1} T^{-C_0} \leq C_0^{-2} T^{-C_0}$. For $C_0 \geq 2$ and $T \geq 2$ it follows that

$$S \geq \frac{1}{2\delta} \geq \frac{C_0^2 T^{C_0}}{2} \geq T^2, \qquad S^{-2} \leq 4\delta^2 \leq \delta, \tag{2.8}$$

where the last inequality uses $\delta \leq 1/4$.

In the applications in Subsection 6.2 and Section 7, the parameter choice (1.14) prescribes ε, and ε^{-1} and T are bounded by powers of the logarithm of the frequency of the preceding wave, while δ is at most a negative power of that frequency times a power of its logarithm. Item 2 of the proof of Theorem 7.1 verifies (2.7) in this way.

For a general background and $E_c = 0$, each coefficient in the proof of the next lemma is a polynomial in ζ, $1/\zeta_3$, $1/m$ and the quantities

$$b_{11},\ b_{22},\ b_{33},\ b_{13},\ b_{31},\ \frac{b_{21}}{S},\ \frac{b_{12}}{S},\ \frac{b_{23}}{S},\ \frac{b_{32}}{S},\ 2Sb_{12},\ 2Sb_{32},\ \frac{\sigma_p}{\sigma},\ S^{-2}. \tag{2.9}$$

The *reference value* of such an expression is its value at $\zeta = \zeta_0$, $2Sb_{12} = 1$, $2Sb_{32} = \varepsilon$ and $\sigma_p/\sigma = 1$, with all other quantities in (2.9) equal to zero. By Lemma 2.3, it is the limit of the expression in the model case as $S \to \infty$.

Lemma 2.4. *Assume (2.7). The scaled components of the wave vector satisfy*

$$\sup|\zeta - \zeta_0| \leq CT^C\delta \leq \tfrac{1}{2}, \qquad \sup\left|\frac{|k|^2\cos^2\varphi}{\lambda^2} - 1\right| \leq CT^C\delta \leq \tfrac{1}{2}. \tag{2.10}$$

In particular

$$\tfrac{1}{2} \leq \zeta_3 \leq \tfrac{3}{2}, \qquad |\zeta| \leq 3T^2. \tag{2.11}$$

The components α_1, α_2 of every amplitude vector satisfy

$$\frac{d}{d\tau}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \left\{-\begin{pmatrix}\varepsilon\tau\sin 2\varphi & \cos^2\varphi(1-\varepsilon\tan\varphi)\\ 1 & 0\end{pmatrix} + \mathcal{E}(\tau)\right\}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \qquad \sup|\mathcal{E}| \leq CT^C\delta,$$

where $\mathcal{E}$ is the same for all amplitude vectors.

Proof. The equations (2.1) and (2.2) are in the time t, and every coefficient in them is a rate of change in t. In this proof primes denote derivatives in τ. Since $d/d\tau = r^{-1}d/dt$, each coefficient is divided by r, which turns B_{ij} into b_{ij} and σ_p into $S\sigma_p/\sigma$. By (2.6) and $r/S = 2\nu$,

$$|b_{ij}| \leq \delta, \qquad |2Sb_{12} - 1| = \left|\frac{B_{12}}{\nu} - 1\right| \leq \delta, \qquad |2Sb_{32} - \varepsilon| = \varepsilon\left|\frac{B_{32}}{\eta} - 1\right| \leq \delta.$$

In the time τ the last equation of (2.2) reads $(\log\sigma_p)' = b_{33}$, and $\sigma_p = \sigma$ at $\tau = 0$. Hence

$$\frac{\sigma_p(\tau)}{\sigma} = \exp\left(\int_0^\tau b_{33}(\tau')\,d\tau'\right),$$

and $\sup|\sigma_p/\sigma - 1| \leq 2T\delta$, because the integral is at most $T\delta \leq 1$ in absolute value.

When $E_c = 0$, the equation for k and (2.2) give

$$\zeta_2' = -b_{22}\zeta_2 - 2Sb_{32}\zeta_3, \qquad \zeta_3' = -b_{33}\zeta_3 - (b_{23} - b_{32})\zeta_2/S,$$
$$\zeta_1' = -b_{11}\zeta_1 - (\sigma_p/\sigma + b_{21}/S + b_{12}/S)\zeta_2 - (b_{31} + b_{13})\zeta_3.$$

In the model case these are the equations of Lemma 2.3. When $E_c \neq 0$, the term $-E_c^T k$ is added to the equation for k, and the frame still evolves over B. In the time τ, the additional coefficient matrix for ζ is

$$-r^{-1}\operatorname{diag}(1,S,1)\,E_c^T\operatorname{diag}(1,S^{-1},1),$$

where E_c^T is the transpose of E_c written in the frame (E_i) of the preceding wave, the matrix with entries $E_j \cdot E_c E_i$. Since $S \geq 1$, its norm is at most $(S/r)|E_c| = |E_c|/(2\nu) \leq \delta$.

Write $\zeta' = (A_0 + A_1)\zeta$, where A_0 is the matrix of the limiting system $\zeta_1' = -\zeta_2$, $\zeta_2' = -\varepsilon\zeta_3$, $\zeta_3' = 0$, and A_1 collects the remaining coefficients. The bounds above give $\sup|A_1| \leq CT\delta$. The vector ζ_0 solves $\zeta_0' = A_0\zeta_0$ with the same initial value as ζ, and $|\zeta_0| \leq 2T^2$. Since $A_0^3 = 0$, its propagator is $\exp(uA_0) = \mathrm{Id} + uA_0 + u^2A_0^2/2$, of norm at most $2T^2$ for $0 \leq u \leq T$. Therefore

$$\zeta(\tau) - \zeta_0(\tau) = \int_0^\tau e^{(\tau-\tau')A_0}A_1(\tau')\zeta(\tau')\,d\tau',$$

so

$$\sup|\zeta - \zeta_0| \leq CT^C\delta\,\bigl(1 + \sup|\zeta - \zeta_0|\bigr).$$

Condition (2.7) gives the first bound of (2.10), hence (2.11) and $m \geq 1/4$. Since $m = \zeta_1^2 + S^{-2}\zeta_2^2 + \zeta_3^2$ and $S^{-2} \leq \delta$ by (2.8), it also gives $|m - \cos^{-2}\varphi| \leq CT^C\delta$. Multiplication by $\cos^2\varphi \leq 1$ proves the second bound.

To express the pressure coefficient in the amplitude equation for $E_c = 0$, put

$$\begin{aligned}
F_1 &= \zeta_1(b_{11} - b_{13}\zeta_1/\zeta_3) + \zeta_2(\sigma_p/\sigma + b_{21}/S - (b_{23}/S)\zeta_1/\zeta_3)\\
&\quad + \zeta_3(b_{31} - b_{33}\zeta_1/\zeta_3),\\
F_2 &= \zeta_1(Sb_{12} - b_{13}\zeta_2/\zeta_3) + \zeta_2(b_{22} - (b_{23}/S)\zeta_2/\zeta_3)\\
&\quad + \zeta_3(Sb_{32} - b_{33}\zeta_2/\zeta_3).
\end{aligned}$$

Then the pressure coefficient of (2.1) satisfies

$$\frac{\lambda S}{r}p_{\mathrm{w}} = \frac{2(F_1\alpha_1 + F_2\alpha_2)}{m}.$$

Substituting for α_3 in the first two amplitude equations gives a system for $(\alpha_1,\alpha_2)^T$ whose coefficient matrix N has entries

$$\begin{aligned}
N_{11} &= -b_{11} + 2b_{13}\zeta_1/\zeta_3 + \boxed{2\zeta_1F_1/m},\\
N_{12} &= \boxed{-2Sb_{12}} + 2b_{13}\zeta_2/\zeta_3 + \boxed{2\zeta_1F_2/m},\\
N_{21} &= \boxed{-\sigma_p/\sigma} - (b_{21} - b_{12})/S + ((b_{23} + b_{32})/S)\zeta_1/\zeta_3 + 2\zeta_2F_1/(S^2m),\\
N_{22} &= -b_{22} + ((b_{23} + b_{32})/S)\zeta_2/\zeta_3 + 2\zeta_2F_2/(S^2m).
\end{aligned}$$

Only the boxed terms have nonzero reference values. Every other term has one of b_{11}, b_{22}, b_{13}, the quotients b_{ij}/S or S^{-2} as a factor, so its reference value is zero. In F_1 only the term $\zeta_2\sigma_p/\sigma$, and in F_2 only ζ_1Sb_{12} and ζ_3Sb_{32}, have nonzero reference values. Hence the reference values of F_1, F_2 and m are $-\varepsilon\tau$, $(\tan\varphi + \varepsilon)/2$ and $\cos^{-2}\varphi$. Thus the reference entries of N are

$$-\varepsilon\tau\sin 2\varphi, \qquad -1 + \tan\varphi(\tan\varphi + \varepsilon)\cos^2\varphi = -\cos^2\varphi(1 - \varepsilon\tan\varphi), \qquad -1, \qquad 0,$$

which are the coefficients of (2.5).

Each entry of N is a polynomial of bounded degree in ζ_1, ζ_2, ζ_3, $1/\zeta_3$, $1/m$ and the quantities (2.9). These are at most CT^2. Each differs from its reference value by at most $CT^C\delta$: the coordinates by (2.10), the reciprocals by a multiple of $|m - \cos^{-2}\varphi| + |\zeta_3 - 1|$, the coefficients by the bounds at the beginning of the proof, and S^{-2} by (2.8). Hence each entry of N differs from its reference value by at most $CT^C\delta$.

For $E_c \neq 0$, the amplitude equation contains the additional term $-(\mathrm{Id} - 2\frac{k}{|k|} \otimes \frac{k}{|k|})E_c\mathbf{a}$, of norm at most $|E_c|\,|\mathbf{a}|$. By (2.3), (2.11) and (2.8), $|\mathbf{a}| \le C|(\alpha_1, \alpha_2)|$, while $\alpha_1 = S\mathbf{a} \cdot E_1$ and $\alpha_2 = \mathbf{a} \cdot E_2$. In the time τ this adds a coefficient matrix of norm at most $C(S/r)|E_c| \le C\delta$ to the reduced amplitude system. □

2.2.2. *The reference solution.* At $\tau = 0$, the reference system (2.5) is $\alpha_1' = -\alpha_2$, $\alpha_2' = -\alpha_1$. Its growing solutions have $\alpha_1 = -\alpha_2$, the direction $-E_1/S + E_2$ of the amplitude in (2.3). As φ increases, the pressure reduces the coefficient driving α_1 from one to $\cos^2\varphi(1 - \varepsilon\tan\varphi)$ and introduces the term $-\varepsilon\tau\sin 2\varphi\,\alpha_1$. The driving coefficient becomes negative when $\varepsilon\tan\varphi > 1$.

The *reference solution* is the solution of the reference system with $(\alpha_1, \alpha_2)(0) = (-1, 1)$, shown in Figure 1. A later wave is started with these data in Subsection 6.2. The second equation of (2.5), $\alpha_2' = -\alpha_1$, holds at every τ. Hence, wherever $\alpha_2 > 0$, the logarithmic derivative of α_2 is

$$Y = \frac{\alpha_2'}{\alpha_2} = -\frac{\alpha_1}{\alpha_2},$$

and the first equation of (2.5) gives the Riccati equation

$$Y' = \cos^2\varphi(1 - \varepsilon\tan\varphi) - \varepsilon\tau\sin 2\varphi\, Y - Y^2, \qquad Y(0) = 1. \tag{2.12}$$

Conversely, if Y solves (2.12), then $(-Y a_{\mathrm{ref}}, a_{\mathrm{ref}})$ is the reference solution, where

$$a_{\mathrm{ref}}(\tau) = \exp\left(\int_0^\tau Y(\tau')\,d\tau'\right).$$

We call a_{ref} the *reference amplitude*.

Lemma 2.5. *Let $0 < \varepsilon \le 1$. The solution Y of (2.12) exists for all $\tau \ge 0$, satisfies $|Y| \le 1$, and has the form*

$$Y = (1 + e)\cos\varphi - \tfrac{1}{4}\varepsilon\tau\sin 2\varphi, \qquad 0 \le e \le \tfrac{1}{8}\varepsilon. \tag{2.13}$$

The reference amplitude satisfies

$$a_{\mathrm{ref}}(\tau) = \sqrt{\cos\varphi(\tau)}\exp\left(\int_0^\tau (1 + e)\cos\varphi\,d\tau'\right). \tag{2.14}$$

The function Y has exactly one zero τ_m, with $2 \le \varepsilon\tau_m \le 3$. It is positive before this zero and negative afterwards, so a_{ref} is largest at τ_m, and

$$\log a_{\mathrm{ref}}(\tau_m) \ge \varepsilon^{-1/2} - 1.$$

Proof. Since $\tan\varphi = \varepsilon\tau^2/2$,

$$\varphi' = \varepsilon\tau\cos^2\varphi, \qquad (\log\cos\varphi)' = -\tfrac{1}{2}\varepsilon\tau\sin 2\varphi.$$

We look for Y in the form (2.13), an ansatz with an unknown function e. To motivate it, write $Y = \cos\varphi + w$, where $\cos\varphi$ is the positive equilibrium of the equation $Y' = \cos^2\varphi - Y^2$ with φ frozen. Neglecting w', w^2, $\varepsilon\tau w\sin 2\varphi$ and the term $-\varepsilon\sin\varphi\cos\varphi$ of order ε, (2.12) reduces to $2w\cos\varphi = -\varepsilon\tau\sin\varphi\cos^2\varphi$, that is, $w = -\frac{1}{4}\varepsilon\tau\sin 2\varphi = \frac{1}{2}(\log\cos\varphi)'$. The factor $1 + e$ absorbs the neglected terms.

Integrating (2.13) and using the formula for $(\log\cos\varphi)'$ gives (2.14). Substituting (2.13) into (2.12) gives

$$e' = \cos\varphi\big(-2e - e^2 + \tfrac{1}{2}\varepsilon\sin\varphi\cos\varphi\big), \qquad e(0) = 0.$$

The right-hand side is nonnegative at $e = 0$ and nonpositive at $e = \varepsilon/8$, because $\sin\varphi\cos\varphi \le 1/2$. Hence e remains in $[0, \varepsilon/8]$ and exists for all $\tau \ge 0$. This constructs the global solution Y.

Wherever $Y \ge 0$, (2.12) gives $Y' \le 1 - Y^2$, because $\cos^2\varphi(1 - \varepsilon\tan\varphi) \le 1$. Since $Y(0) = 1$, $Y(\tau)$ never exceeds one. Also $\sin\varphi\cos\varphi \le \min\{1/2, \cot\varphi\}$, so for $\tau > 0$,

$$\tfrac{1}{4}\varepsilon\tau\sin 2\varphi \le \min\{\varepsilon\tau/4, 1/\tau\} \le \tfrac{1}{2}.$$

Equation (2.13) therefore gives $Y \geq -1/2$.

Write $Y = \cos\varphi(1 + e - \frac{1}{2}\varepsilon\tau\sin\varphi)$. It is positive for $\varepsilon\tau \leq 2$, because $e \geq 0$ and $\sin\varphi < 1$. For $\varepsilon\tau = 3$, we have $\tan\varphi = 9/(2\varepsilon) \geq 9/2$, hence $\sin\varphi > 3/4$ and $\frac{1}{2}\varepsilon\tau\sin\varphi > 9/8 \geq 1 + e$. Thus Y is negative there. At any zero with $\varepsilon\tau \geq 2$,

$$Y' = \cos^2\varphi(1 - \varepsilon\tan\varphi) = \cos^2\varphi\big(1 - (\varepsilon\tau)^2/2\big) < 0.$$

Consequently Y has exactly one zero τ_m, with $2 \leq \varepsilon\tau_m \leq 3$, and has the stated signs on its two sides.

For $0 \leq \tau \leq (2/\varepsilon)^{1/2}$, we have $\tan\varphi \leq 1$, hence $\cos\varphi \geq 2^{-1/2}$. This interval ends before $\tau_m \geq 2/\varepsilon$, where a_{ref} is largest. By (2.14) and $e \geq 0$,

$$\log a_{\mathrm{ref}}(\tau_m) \geq \log a_{\mathrm{ref}}\big((2/\varepsilon)^{1/2}\big) \geq 2^{-1/2}(2/\varepsilon)^{1/2} - \tfrac{1}{4}\log 2 \geq \varepsilon^{-1/2} - 1. \qquad \square$$

Relative to the reference solution, the reference system is triangular. To compare a solution (α_1, α_2) of (2.5) with the reference solution $(-Ya_{\mathrm{ref}}, a_{\mathrm{ref}})$, we divide by the reference amplitude, which removes the exponential growth common to the growing solutions, and we separate the size of α_2 from the deviation of the ratio α_1/α_2 from its reference value $-Y$. We call

$$h = \frac{\alpha_2}{a_{\mathrm{ref}}}, \qquad l = -\frac{\alpha_1 + Y\alpha_2}{a_{\mathrm{ref}}\cos\varphi} \tag{2.15}$$

the *normalized coordinates* of (α_1, α_2). The factor $1/\cos\varphi$ in l gives the equations for h and l below the same factor $\cos\varphi$, which yields a propagator bound uniform in ε and in the length of the interval. By the second equation of (2.5) and $a'_{\mathrm{ref}} = Ya_{\mathrm{ref}}$, we have $h' = l\cos\varphi$. The trace of the reference coefficient matrix is $-\varepsilon\tau\sin 2\varphi = 2(\log\cos\varphi)'$. By Liouville's formula, the determinant

$$\det\begin{pmatrix}\alpha_1 & -Ya_{\mathrm{ref}}\\ \alpha_2 & a_{\mathrm{ref}}\end{pmatrix} = a_{\mathrm{ref}}(\alpha_1 + Y\alpha_2) = -a_{\mathrm{ref}}^2 l\cos\varphi$$

is a constant multiple of $\cos^2\varphi$. Formula (2.14) thus gives

$$l(\tau) = l(0)\exp\left(-2\int_0^\tau (1+e)\cos\varphi\, d\tau'\right).$$

The reference system becomes

$$h' = l\cos\varphi, \qquad l' = -2(1+e)l\cos\varphi, \tag{2.16}$$

and the reference solution is $(h, l) = (1, 0)$. The second equation shows that $|l|$ does not increase. Since $e \geq 0$, for $\tau_1 \leq \tau_2$,

$$|h(\tau_2) - h(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |l|\cos\varphi \leq \int_{\tau_1}^{\tau_2} (1+e)|l|\cos\varphi = \tfrac{1}{2}\big(|l(\tau_1)| - |l(\tau_2)|\big).$$

The triangle inequality and $|l(\tau_2)| \leq |l(\tau_1)|$ give

$$|h(\tau_2)| + |l(\tau_2)| \leq |h(\tau_1)| + |l(\tau_1)|.$$

Since $|(h,l)| \leq |h| + |l| \leq \sqrt{2}\,|(h,l)|$, the propagator $\Psi_0(\tau_2, \tau_1)$ of (2.16) has norm at most 2 for $\tau_1 \leq \tau_2$, uniformly in both times and in ε.

2.2.3. *Comparison with the reference solution.* The next lemma shows that the wave follows the reference solution on $0 \leq \tau \leq T$. Dividing by a_{ref} removes the exponential growth, and the bound for Ψ_0 gives errors polynomial in T. A direct Gronwall estimate for (α_1, α_2) would introduce $\exp(CT)$, while (2.7) controls polynomial losses in T. The relative bounds give $|\mathbf{a}|$ up to a factor $\exp(\pm\varepsilon)$, so the wave inherits the reference gain. Recovering the normalized coordinates from an amplitude vector costs a factor S, because its E_1 component is scaled by S^{-1}.

Lemma 2.6. *Assume (2.7), and let* $\mathbf{a}$ *be the amplitude vector with* $\mathbf{a}(t_0) = -E_1(t_0)/S + E_2(t_0)$, *that is, with* $(\alpha_1, \alpha_2)(0) = (-1, 1)$ *in (2.3), the initial data of the reference solution. Then* $\alpha_2 > 0$ *on* $0 \le \tau \le T$, *and*

$$\left|\frac{\alpha_1}{\alpha_2} + Y\right| \le CT^C\delta, \qquad \alpha_2 \le |\mathbf{a}| \le (1 + CT^C\delta)\alpha_2. \tag{2.17}$$

Moreover,

$$\left|\log \frac{|\mathbf{a}|}{a_{\mathrm{ref}}}\right| \le \varepsilon. \tag{2.18}$$

For $0 \le \tau_1 \le \tau_2 \le T$, *let* $\Phi(\tau_2, \tau_1)$ *map* $\mathbf{b}(\tau_1)$ *to* $\mathbf{b}(\tau_2)$ *for every amplitude vector* $\mathbf{b}$. *It maps the plane perpendicular to* $k(\tau_1)$ *onto the plane perpendicular to* $k(\tau_2)$, *and*

$$\|\Phi(\tau_2, \tau_1)\| \le CS\,\frac{|\mathbf{a}(\tau_2)|}{|\mathbf{a}(\tau_1)|}.$$

Proof. For an amplitude vector $\mathbf{b}$, let $\beta = (\beta_1, \beta_2, \beta_3)$ be its scaled components, defined as those of $\mathbf{a}$ in (2.3):

$$\mathbf{b} = \frac{\beta_1}{S}E_1 + \beta_2 E_2 + \frac{\beta_3}{S}E_3.$$

Its normalized coordinates are given by (2.15) with (β_1, β_2) in place of (α_1, α_2), that is,

$$\begin{pmatrix} h \\ l \end{pmatrix} = \frac{1}{a_{\mathrm{ref}}} R \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 \\ -(\cos\varphi)^{-1} & -Y(\cos\varphi)^{-1} \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} -Y & -\cos\varphi \\ 1 & 0 \end{pmatrix}.$$

By Lemma 2.4, (h, l) solves (2.16) with the additional coefficient matrix $R\mathcal{E}R^{-1}$. Since $|Y| \le 1$ and $(\cos\varphi)^{-1} \le T^2$, we have $|R| \le CT^2$, $|R^{-1}| \le C$ and $|R\mathcal{E}R^{-1}| \le CT^C\delta$.

Let $\Psi(\tau_2, \tau_1)$ be the propagator of this perturbed system. Variation of constants gives

$$\Psi(\tau_2, \tau_1) = \Psi_0(\tau_2, \tau_1) + \int_{\tau_1}^{\tau_2} \Psi_0(\tau_2, \tau')(R\mathcal{E}R^{-1})(\tau')\Psi(\tau', \tau_1)\,d\tau'.$$

Consequently $\sup|\Psi| \le 2 + 2T \cdot CT^C\delta \sup|\Psi|$, and (2.7) gives $|\Psi(\tau_2, \tau_1)| \le 4$.

For the prescribed initial amplitude $\mathbf{a}(t_0)$ the normalized coordinates at $\tau = 0$ are $(h, l)(0) = (1, 0)$, a constant solution of the reference system. Applying the preceding formula to this vector gives

$$\sup|(h - 1, l)| \le 2T \cdot CT^C\delta \cdot 4 \le CT^C\delta \le \tfrac{1}{2}.$$

In particular $h \ge 1/2$, and the coordinate identities give

$$\alpha_2 = a_{\mathrm{ref}} h > 0, \qquad \frac{\alpha_1}{\alpha_2} + Y = -\frac{l\cos\varphi}{h}, \qquad \left|\frac{\alpha_1}{\alpha_2} + Y\right| \le 2|l| \le CT^C\delta.$$

This is the first bound of (2.17), and $|Y| \le 1$ gives $|\alpha_1| \le 2\alpha_2$. Since $\alpha_3 = -(\zeta_1\alpha_1 + \zeta_2\alpha_2)/\zeta_3$, the bounds (2.11) give $|\alpha_3| \le 18T^2\alpha_2$. By (2.3),

$$|\mathbf{a}|^2 = \alpha_2^2 + S^{-2}(\alpha_1^2 + \alpha_3^2).$$

Since $S^{-2} \le \delta$ by (2.8), this proves the second bound of (2.17). Thus $\log(|\mathbf{a}|/a_{\mathrm{ref}}) = \log h + \log(|\mathbf{a}|/\alpha_2)$ is at most $CT^C\delta$ in absolute value, and (2.7) gives (2.18).

To bound $\Phi(\tau_2, \tau_1)$, note that for an arbitrary amplitude vector $\mathbf{b}$, $\beta_1 = -a_{\mathrm{ref}}(Yh + l\cos\varphi)$, $\beta_2 = a_{\mathrm{ref}}h$ and $\beta_3 = -(\zeta_1\beta_1 + \zeta_2\beta_2)/\zeta_3$. Hence $\mathbf{b}/a_{\mathrm{ref}} = hL_h + lL_l$, where

$$L_h = -\frac{Y}{S}E_1 + E_2 + \frac{\zeta_1 Y - \zeta_2}{S\zeta_3}E_3, \qquad L_l = \frac{\cos\varphi}{S}\left(-E_1 + \frac{\zeta_1}{\zeta_3}E_3\right).$$

By (2.11) and $S \ge T^2$, $|L_h| + |L_l| \le C$. Conversely, $h = (\mathbf{b}/a_{\mathrm{ref}}) \cdot E_2$, and $lL_l = \mathbf{b}/a_{\mathrm{ref}} - hL_h$. By (2.10),

$$|L_l| = \frac{\cos\varphi}{S}\sqrt{1 + \zeta_1^2/\zeta_3^2} \ge \frac{c}{S}, \qquad |h| + |l| \le CS|\mathbf{b}|/a_{\mathrm{ref}}.$$

Together with $|\Psi| \le 4$, these two coordinate bounds give

$$\|\Phi(\tau_2,\tau_1)\| \le CS\,\frac{a_{\mathrm{ref}}(\tau_2)}{a_{\mathrm{ref}}(\tau_1)} \le CS\,\frac{|\mathbf{a}(\tau_2)|}{|\mathbf{a}(\tau_1)|},$$

where the last step uses (2.18). □

2.2.4. *The couplings of the new wave.* Recall that the new wave evolves over the gradient $M = B + \sigma_p E_2 \otimes E_1 + E_c$ of Subsection 2.2.1. The couplings ν_+, η_+ of M in the frame of the new wave drive its successor. The first controls the sign of the pressure coefficient, since

$$p_{\mathrm{w}} = \frac{2k\cdot M\mathbf{a}}{|k|^2} = 2\nu_+\frac{|\mathbf{a}|}{|k|}.$$

Subsection 5.1 uses this positivity throughout the amplification interval.

Lemma 2.7. *Assume (2.7), and let* $\mathbf{a}$ *be the amplitude vector of Lemma 2.6. Put* $E_{1,+} = k/|k|$, $E_{2,+} = \mathbf{a}/|\mathbf{a}|$ *and* $E_{3,+} = E_{1,+}\times E_{2,+}$, *and define*

$$\nu_+ = E_{1,+}\cdot ME_{2,+}, \qquad \eta_+ = E_{3,+}\cdot ME_{2,+}.$$

Then $\nu_+ \ge \varepsilon\nu/2$ *on* $0 \le \tau \le T$. *Where* $\varepsilon\tan\varphi \ge 1$, *that is, for* $\sqrt{2}/\varepsilon \le \tau \le T$,

$$\left|\frac{\nu_+}{\nu} - 1\right| \le \varepsilon, \qquad \left|\frac{\eta_+/\nu_+}{\cos\varphi} - 1\right| \le \frac{1}{2}. \tag{2.19}$$

Proof. Put

$$q = \frac{\alpha_1}{\alpha_2}, \qquad z = \frac{\alpha_3}{\alpha_2} = -\frac{\zeta_1 q + \zeta_2}{\zeta_3}, \qquad n = \frac{|\mathbf{a}|}{\alpha_2}.$$

Then

$$\frac{\mathbf{a}}{\alpha_2} = \frac{q}{S}E_1 + E_2 + \frac{z}{S}E_3, \qquad n^2 = 1 + \frac{q^2+z^2}{S^2}.$$

For $E_c = 0$, the components of $M\mathbf{a}/(r\alpha_2)$ in the frame (E_1,E_2,E_3) of the preceding wave are $(f_1/S, f_2, f_3/S)$, where

$$f_1 = b_{11}q + Sb_{12} + b_{13}z, \qquad f_2 = (\sigma_p/\sigma + b_{21}/S)q + b_{22} + (b_{23}/S)z, \qquad f_3 = b_{31}q + Sb_{32} + b_{33}z.$$

In the same frame, the components of $(k/\lambda)\times(\mathbf{a}/\alpha_2)$ are

$$\left(\frac{\zeta_2 z}{S^2} - \zeta_3, \frac{\zeta_3 q - \zeta_1 z}{S}, \zeta_1 - \frac{\zeta_2 q}{S^2}\right).$$

Since $E_{1,+} = k/|k|$, $E_{2,+} = \mathbf{a}/|\mathbf{a}|$ and $E_{3,+} = (k\times\mathbf{a})/(|k||\mathbf{a}|)$,

$$\nu_+ = \frac{k\cdot M\mathbf{a}}{|k||\mathbf{a}|}, \qquad \eta_+ = \frac{(k\times\mathbf{a})\cdot M\mathbf{a}}{|k||\mathbf{a}|^2}.$$

We compute these scalar products in the frame of the preceding wave, pairing k/λ and $(k/\lambda)\times(\mathbf{a}/\alpha_2)$ with $M\mathbf{a}/(r\alpha_2)$. With $|k| = \lambda\sqrt{m}$, $|\mathbf{a}| = n\alpha_2$ and $r/S = 2\nu$ this gives

$$\nu_+/\nu = \frac{2(\zeta_1 f_1 + \zeta_2 f_2 + \zeta_3 f_3)}{\sqrt{m}\,n}, \tag{2.20}$$

$$\eta_+/\nu = \frac{2}{\sqrt{m}\,n^2}\left\{\left(-\zeta_3 + \frac{\zeta_2 z}{S^2}\right)f_1 + (\zeta_3 q - \zeta_1 z)f_2 + \left(\zeta_1 - \frac{\zeta_2 q}{S^2}\right)f_3\right\}. \tag{2.21}$$

The matrix E_c adds $E_{1,+}\cdot E_cE_{2,+}/\nu$ to ν_+/ν and $E_{3,+}\cdot E_cE_{2,+}/\nu$ to η_+/ν. Each is at most δ in absolute value.

At the reference values,

$$\zeta = (\tan\varphi, -\varepsilon\tau, 1), \qquad q = -Y, \qquad z = Y\tan\varphi + \varepsilon\tau,$$

and $f_1 = 1/2$, $f_2 = -Y$, $f_3 = \varepsilon/2$, $n = 1$ and $m = \cos^{-2}\varphi$. The two expressions in (2.20)–(2.21) are therefore, respectively,

$$\sin\varphi + \varepsilon\cos\varphi + 2\varepsilon\tau Y\cos\varphi, \qquad (\varepsilon\tan\varphi - 1)\cos\varphi + \frac{2Y^2}{\cos\varphi} + 2\varepsilon\tau Y\sin\varphi.$$

In the second expression the terms $\varepsilon\tan\varphi\cos\varphi$, $2Y^2/\cos\varphi$ and $2\varepsilon\tau Y\sin\varphi$ are large compared with $\cos\varphi$ when $\varepsilon\tan\varphi$ is large, and $\varepsilon\tan\varphi = \frac{1}{2}(\varepsilon\tau)^2$ grows with τ. Their large parts cancel. To see this, write $Y = (1 + e - u)\cos\varphi$ by (2.13), with $u = \frac{1}{2}\varepsilon\tau\sin\varphi$, so that $2\varepsilon\tau\sin\varphi = 4u$ and $2u^2 = \varepsilon\tan\varphi\sin^2\varphi$. Dividing the second expression by $\cos\varphi$ gives

$$\begin{aligned}&(\varepsilon\tan\varphi - 1) + 2(1 + e - u)^2 + 4u(1 + e - u)\\ &\quad= \varepsilon\tan\varphi - 1 + \big[2(1+e)^2 - 4u(1+e) + 2u^2\big] + \big[4u(1+e) - 4u^2\big]\\ &\quad= 1 + 4e + 2e^2 + \big(\varepsilon\tan\varphi - 2u^2\big).\end{aligned}$$

The terms $\pm 4u(1+e)$ cancel, and the large parts of $\varepsilon\tan\varphi$ and $2u^2$ cancel:

$$\varepsilon\tan\varphi - 2u^2 = \varepsilon\tan\varphi\,(1 - \sin^2\varphi) = \varepsilon\sin\varphi\cos\varphi \le \tfrac{1}{2}\varepsilon.$$

Similarly, $2\varepsilon\tau Y\cos\varphi = 2(1+e)\varepsilon\tau\cos^2\varphi - 2\varepsilon\sin^2\varphi\cos\varphi$. Thus the reference values of ν_+/ν and η_+/ν are

$$\begin{aligned}&\sin\varphi + \varepsilon\cos\varphi\cos 2\varphi + 2(1+e)\varepsilon\tau\cos^2\varphi,\\ &(1 + 4e + 2e^2 + \varepsilon\sin\varphi\cos\varphi)\cos\varphi.\end{aligned} \tag{2.22}$$

Where $\varepsilon\tan\varphi \ge 1$, we have $\cos\varphi \le \cot\varphi \le \varepsilon$, so $1 - \varepsilon^2 \le \sin\varphi \le 1$. Also $0 \le e \le \varepsilon/8$ and

$$\varepsilon\tau\cos^2\varphi \le \varepsilon\tau\cot^2\varphi = \frac{4}{\varepsilon\tau^3} \le \sqrt{2}\,\varepsilon^2.$$

The two reference values in (2.22) are $1 + O(\varepsilon^2)$ and $(1 + O(\varepsilon))\cos\varphi$.

Where $\varepsilon\tan\varphi \le 1$, we have $\varepsilon\tau \le \sqrt{2}$, hence $Y > 0$ by Lemma 2.5. The unsimplified reference value of ν_+/ν is at least $\sin\varphi + \varepsilon\cos\varphi \ge \varepsilon$.

For the wave, in (2.20)–(2.21), the coefficients differ from their reference values by at most $CT\delta$; ζ differs from ζ_0 by at most $CT^C\delta$; the terms with S^{-2} in those formulas, in m and in n, are at most $CT^C\delta$; and q differs from $-Y$ by at most $CT^C\delta$, by (2.17). The formula for z and the bound $\zeta_3 \ge 1/2$ give the same bound for its change. The numerators are polynomials of bounded degree in these quantities and $1/\zeta_3$, which are at most CT^2, while the denominators $\sqrt{m}\,n$ and $\sqrt{m}\,n^2$ are at least $1/2$. Together with the contribution of E_c, the two ratios therefore differ from their reference values by at most $CT^C\delta$, which is at most $\varepsilon T^{-2}/8$ by (2.7).

In both ranges the reference value of ν_+/ν is at least ε; in the late range this uses $C_0\varepsilon \le 1$. Hence $\nu_+/\nu \ge \varepsilon/2$ on the whole interval. In the late range, $\cos\varphi \ge T^{-2}$. Combining the perturbation bounds with (2.22) gives

$$\left|\frac{\nu_+}{\nu} - 1\right| \le \varepsilon, \qquad \frac{\eta_+}{\nu_+} = (1 + O(\varepsilon))\cos\varphi.$$

The latter gives (2.19) for C_0 sufficiently large. □

2.2.5. *The crossing.* After the peak of the reference solution at τ_m, the new wave becomes the shear for the next wave, whose coupling ratio ε_+ is prescribed in advance. By (2.19), the coupling ratio η_+/ν_+ of the new wave is comparable to $\cos\varphi$. It therefore reaches ε_+ when $\cos\varphi$ is comparable to ε_+, that is, when $\tan\varphi$ is comparable to $1/\varepsilon_+$, or τ to $(\varepsilon\varepsilon_+)^{-1/2}$. We call a time τ at which $\eta_+/\nu_+ = \varepsilon_+$ a *crossing*: there the ratio reaches the threshold ε_+ prescribed for the next wave. The next wave is activated at a crossing. The reference amplitude decreases after τ_m, and the wave-vector length is comparable to $\lambda/\cos\varphi$. The next proposition bounds their product for the actual wave at a crossing, and shows that its amplitude is small compared with its maximum during the first unit

of normalized time. The strength $\sigma_+ = |\mathbf{a}||k|$ at the crossing and the coupling ν_+ supply the frozen parameters σ, ν for the next wave.

Proposition 2.8. *Assume (2.7), let* $\mathbf{a}$ *be the amplitude vector of Lemma 2.6, with the couplings of Lemma 2.7, and suppose that* $0 < \varepsilon_+ \le \varepsilon/32$ *and* $T \ge 2(\varepsilon\varepsilon_+)^{-1/2}$. *Then the interval* $\tau_m \le \tau \le 2(\varepsilon\varepsilon_+)^{-1/2}$, *where* $\tau_m \ge 2/\varepsilon$ *is the zero of* Y *(Lemma 2.5), contains a crossing. At every crossing in this interval the strength* $\sigma_+ = |\mathbf{a}(\tau)||k(\tau)|$ *satisfies*

$$C^{-1}(\varepsilon\varepsilon_+)^{-1/2} \le \frac{\sigma_+}{\lambda \max_{[0,T]} |\mathbf{a}|} \le C(\varepsilon\varepsilon_+)^{-1/2}. \tag{2.23}$$

Moreover

$$|\mathbf{a}(\tau)| \le \exp\bigl(3 - \varepsilon^{-1/2}\bigr) \max_{[0,T]} |\mathbf{a}| \qquad (0 \le \tau \le 1). \tag{2.24}$$

Proof. By Lemma 2.5, $2/\varepsilon \le \tau_m \le 3/\varepsilon$. Since $\varepsilon_+ \le \varepsilon/32$, we have $(\varepsilon\varepsilon_+)^{-1/2} \ge 4/\varepsilon$. Thus $\tau_m < 2(\varepsilon\varepsilon_+)^{-1/2} \le T$, and $[\tau_m, T]$ lies in the range $\tau \ge \sqrt{2}/\varepsilon$ of (2.19).

For $\tau_m \le \tau \le T$, we have $\tan\varphi = \varepsilon\tau^2/2 \ge 2/\varepsilon \ge 1$, hence

$$\frac{1}{2\tan\varphi} \le \cos\varphi \le \frac{1}{\tan\varphi}, \qquad \frac{1}{4\tan\varphi} \le \frac{\eta_+}{\nu_+} \le \frac{3}{2\tan\varphi},$$

by (2.19). At τ_m, $\tan\varphi \le 9/(2\varepsilon)$, so $\eta_+/\nu_+ \ge \varepsilon/18 > \varepsilon_+$. At $\tau = 2(\varepsilon\varepsilon_+)^{-1/2}$, we have $\tan\varphi = 2/\varepsilon_+$, so $\eta_+/\nu_+ \le 3\varepsilon_+/4 < \varepsilon_+$. Continuity therefore gives a crossing in this interval. At every crossing in $[\tau_m, T]$ the displayed bounds give $1/(4\varepsilon_+) \le \tan\varphi \le 3/(2\varepsilon_+)$, so $\cos\varphi$ is comparable to ε_+.

To estimate the strength at a crossing, we compare the amplitude there with its maximum on $[0,T]$. By (2.18), $|\mathbf{a}|$ lies between $\exp(-\varepsilon)a_{\mathrm{ref}}$ and $\exp(\varepsilon)a_{\mathrm{ref}}$. The maximum of a_{ref} on $[0,T]$ is attained at τ_m, so for $\tau_m \le \tau \le T$, $|\mathbf{a}(\tau)|/\max_{[0,T]}|\mathbf{a}|$ is bounded above and below by absolute multiples of $a_{\mathrm{ref}}(\tau)/a_{\mathrm{ref}}(\tau_m)$. Formula (2.14) gives

$$\frac{a_{\mathrm{ref}}(\tau)}{a_{\mathrm{ref}}(\tau_m)} = \left(\frac{\cos\varphi(\tau)}{\cos\varphi(\tau_m)}\right)^{1/2} \exp\left(\int_{\tau_m}^{\tau} (1+e)\cos\varphi\, d\tau'\right).$$

The integral lies between zero and two, since $\int_{\tau_m}^{\infty} \cos\varphi\, d\tau \le 2/(\varepsilon\tau_m) \le 1$. Also $|k|$ is comparable to $\lambda/\cos\varphi$, by (2.10). Thus $\sigma_+/(\lambda\max_{[0,T]}|\mathbf{a}|)$ is comparable to $(\cos\varphi(\tau)\cos\varphi(\tau_m))^{-1/2}$. Here $\cos\varphi(\tau_m)$ is comparable to ε, because $2/\varepsilon \le \tan\varphi(\tau_m) \le 9/(2\varepsilon)$, and at a crossing $\cos\varphi$ is comparable to ε_+. This proves (2.23).

For $0 \le \tau \le 1$, the bound $Y \le 1$ gives $a_{\mathrm{ref}}(\tau) \le \exp(\tau) \le \exp(1)$, while $\log a_{\mathrm{ref}}(\tau_m) \ge \varepsilon^{-1/2} - 1$ by Lemma 2.5. By (2.18) at such τ and at τ_m, $|\mathbf{a}(\tau)| \le \exp(1+\varepsilon)$ and $|\mathbf{a}(\tau_m)| \ge \exp(\varepsilon^{-1/2} - 1 - \varepsilon)$. Since $2\varepsilon \le 1$, this gives (2.24). □

3. Localized waves over a smooth parent

We transport the phase and envelope of a Craik–Criminale wave by the parent flow. The amplitude evolves along each particle path as in the affine model. Its variation from particle to particle, together with the spatial cutoff, requires a correction to keep the velocity divergence free.

3.1. Lagrangian coordinates and the leading wave.

Let $I = [0,T]$, $T > 0$, and let U be a smooth odd divergence-free velocity on $I \times \mathbb{T}^3$. We need only its flow X and its velocity gradient. Write a for a particle's position at time zero and $x = X(t,a)$ for its position at time t. Set

$$X(0,a) = a, \qquad M = (D_x U) \circ X, \qquad \mathbf{d} = (D_a X)^{-T} \nabla_a.$$

Time derivatives in these coordinates hold a fixed. On the lift to $\mathbb{R}^3$, $X(t, a+n) = X(t,a) + n$ for $n \in \mathbb{Z}^3$.

Fix the time-zero wave vector $k_* \in \mathbb{R}^3 \setminus \{0\}$ and give the particle with label a the phase $\phi(a) = k_* \cdot a$. The transported phase $\phi \circ X(t,\cdot)^{-1}$ has Eulerian gradient $\kappa = \mathbf{d}\phi = (D_a X)^{-T} k_*$ at $X(t,a)$. For a

flow linear in a the transported phase is $\kappa \cdot x$, as in Lemma 2.1. A spatial derivative of a wave can fall on its amplitude or on its oscillatory factor. To distinguish the two, write a profile as $f(t, a, s)$ with an independent phase $s \in \mathbb{R}/\mathbb{Z}$, measured in cycles, and set

$$f^\natural(t, X(t, a)) = f(t, a, \phi(a)), \qquad \widetilde{\mathbf{d}} = \mathbf{d} + \kappa \partial_s.$$

Here $\mathbf{d}$ differentiates at fixed s, and $\kappa\partial_s$ differentiates through the phase. Along the parent flow the phase is constant, so the two chain rules are

$$\nabla_x f^\natural = (\widetilde{\mathbf{d}} f)^\natural, \qquad (\partial_t + U \cdot \nabla_x) f^\natural = (f_t)^\natural. \tag{3.1}$$

The phase modulo one is periodic in a exactly when $k_* \in \mathbb{Z}^3$. To allow arbitrary k_*, we evaluate the oscillatory profiles on the lifted label cell $[-1/2, 1/2]^3$ and extend periodically. They vanish near the cell boundary, so the extension is smooth. Phase-independent profiles are already periodic in a.

Along each particle path the local velocity gradient M plays the role of the matrix D in (2.1). Indeed, differentiating $\kappa = (D_a X)^{-T} k_*$ and using $\partial_t D_a X = M D_a X$ gives $\kappa_t = -M^T \kappa$. Choose a smooth envelope $g(a)$ supported in the interior of the label cell and an *activation time* $t_0 \in [0, T)$. On a neighborhood of $\operatorname{supp} g$ the amplitude, called the *seed*, solves the other Craik–Criminale equation,

$$\mathbf{a}_t = -M\mathbf{a} + 2\kappa \frac{\kappa \cdot M\mathbf{a}}{|\kappa|^2}, \qquad \mathbf{a}(t_0, a) = \mathbf{a}_*(a), \qquad \kappa(t_0, a) \cdot \mathbf{a}_*(a) = 0. \tag{3.2}$$

The initial amplitude is smooth and may depend on a. For the first wave it is constant and $t_0 = 0$. By (2.1), the seed remains perpendicular to κ for $t_0 \le t \le T$.

Let F_0 be a smooth odd 1-periodic profile and P_0 its zero-mean periodic primitive. The *leading oscillation* and its pressure are

$$A_1 = g\mathbf{a}F_0, \qquad \mathsf{p}[A_1] = -p_{\mathrm{w}} g P_0, \qquad p_{\mathrm{w}} = 2\frac{\kappa \cdot M\mathbf{a}}{|\kappa|^2}. \tag{3.3}$$

Along the particle with label a the leading oscillation is $A_1^\natural(t, X(t, a)) = g(a)\mathbf{a}(t, a)F_0(\phi(a))$. The pressure balances the acceleration along κ. Multiplying (3.2) by gF_0 gives $A_{1,t} + MA_1 = p_{\mathrm{w}}\kappa g F_0$, and $P_0' = F_0$ gives

$$A_{1,t} + MA_1 + \kappa \partial_s \mathsf{p}[A_1] = 0. \tag{3.4}$$

The seed is transverse, so the phase derivative contributes zero to $\nabla_x \cdot A_1^\natural$. Differentiating $g\mathbf{a}$ still leaves $(\mathbf{d} \cdot A_1)^\natural$. To cancel this divergence while keeping the wave localized, we look for a potential whose curl has fast part A_1. By (3.1), the Eulerian curl of $Q^\natural$ is $(\widetilde{\mathbf{d}} \times Q)^\natural$. We need the same correction for each later oscillation. For a profile V with $\kappa \cdot V = 0$ and phase mean $\langle V \rangle_s = 0$, the fast part of the curl is $\kappa \times \partial_s Q$. To make it equal to V, take

$$Q[V] = -\partial_s^{-1} \frac{\kappa \times V}{|\kappa|^2}, \qquad C[V] = \mathbf{d} \times Q[V], \tag{3.5}$$

where ∂_s^{-1} is the zero-mean periodic primitive in s. Since $\kappa \cdot V = 0$, the vector triple product gives $\kappa \times \partial_s Q[V] = V$. The other part of the curl is $C[V]$, the *divergence correction*. Its component along κ determines how it differentiates the phase of another oscillation. The identities for the completed velocity and for this component are

$$V + C[V] = \widetilde{\mathbf{d}} \times Q[V], \qquad \kappa \cdot C[V] = -\partial_s^{-1}(\mathbf{d} \cdot V). \tag{3.6}$$

For the second identity, apply $\widetilde{\mathbf{d}} \cdot$ to the first. The $\mathbf{d}_i$ commute as pulled-back coordinate derivatives. Since $\kappa_j = \mathbf{d}_j \phi$ and κ is independent of s, the $\widetilde{\mathbf{d}}_i$ commute as well. Thus both $\widetilde{\mathbf{d}} \cdot (\widetilde{\mathbf{d}} \times Q[V])$ and $\mathbf{d} \cdot C[V]$ vanish. Transversality leaves $\partial_s(\kappa \cdot C[V]) = -\mathbf{d} \cdot V$. Taking the zero-mean primitive gives (3.6), since $Q[V]$ and $C[V]$ have zero phase mean.

For the leading oscillation the potential is

$$Q[A_1] = gP_0 \frac{\mathbf{a} \times \kappa}{|\kappa|^2}.$$

The potential supplies a factor $|\kappa|^{-1}$, and its curl at fixed phase costs one label derivative, as quantified in (5.31). Both $Q[V]$ and $C[V]$ vanish outside the label support of V at each time, since the primitive acts at fixed label and the curl is local in a. Thus the corrected velocity $(A_1 + C[A_1])^\natural$ is a curl supported in $X(t, \operatorname{supp} g)$. The leading pressure $\mathsf{p}[A_1]^\natural$ also vanishes outside this set. In the construction g is even and equals one near zero, and $\mathbf{a}_*$ is even in a. The odd flow gives even M and κ, so the seed is even in a. Since F_0 is odd and P_0 is even, the completed Eulerian velocity is odd.

A later wave must switch on at its activation time. Take a smooth time cutoff $0 \le \widetilde{\chi} \le 1$ on $[0, T]$ vanishing for $t < t_0$, with $\widetilde{\chi} \equiv 1$ for the first wave. Set

$$V_1 = \widetilde{\chi} A_1, \qquad \mathsf{p}_1 = \widetilde{\chi} \mathsf{p}[A_1], \qquad C_2 = \widetilde{\chi} C[A_1] = C[V_1]. \tag{3.7}$$

For $t_0 > 0$ these products extend smoothly by zero across t_0 because the cutoff is flat there. Multiplication by a time cutoff preserves the spatial divergence cancellation. Its derivative along the parent flow contributes the term

$$R_{\widetilde{\chi}} = \widetilde{\chi}' \, (A_1 + C[A_1]) \tag{3.8}$$

to the residual, so the cutoff error depends on the size of the wave while it is being switched on.

3.2. **The profile.** We choose $F_0 = F_h$ to prescribe the central velocity gradient and control the wave pressure. Since $g(0) = 1$ and $F_h(0) = 0$, taking $F_h'(0) = 1$ gives the leading oscillation $A_1^\natural$ the gradient $\mathbf{a} \otimes \kappa$ at the origin. In the affine model the wave pressure has Hessian $-p_{\mathrm{w}} F_h'(k \cdot x) k \otimes k$. When $p_{\mathrm{w}} > 0$, the bound $F_h' \ge -Ch$ controls its largest eigenvalue, as in (1.11).

Lemma 3.1. *There are odd smooth 1-periodic functions F_h, $0 < h \le 1$, with zero-mean primitives P_h, such that*

$$F_h'(0) = 1, \qquad -Ch \le F_h' \le 1, \qquad \|F_h\|_\infty \le Ch, \qquad \left\|\partial_s^j F_h\right\|_\infty \le C_j h^{1-j} \quad (j \ge 2).$$

Proof. Fix an even $\psi \in C_c^\infty((-1/4, 1/4))$ with $0 \le \psi \le 1$ and $\psi(0) = 1$. The bump $\psi(s/h)$ has integral $c_h = h \int_{\mathbb{R}} \psi < h/2$ over one period. Subtract its mean to obtain a periodic primitive, and normalize the slope at zero by setting

$$F_h'(s) = \frac{\psi(s/h) - c_h}{1 - c_h} \quad (|s| \le 1/2), \qquad F_h(0) = 0.$$

The derivative is even with mean zero, so F_h is odd and periodic and has a zero-mean periodic primitive P_h. Since $c_h < 1/2$, the slope equals one at zero and lies between $-2c_h$ and one. Integrating the lower bound from 0 to s and from s to 1, using $F_h(0) = F_h(1) = 0$, gives $\|F_h\|_\infty \le 2c_h$. Differentiating $\psi(s/h)$ another $j - 1$ times gives the bound $C_j h^{1-j}$. □

4. Selecting initial data by the endpoint problem

For a prescribed force f in the linearized Euler equation (1.6), we choose the initial value $b(0)$ of the phase-independent correction to preserve each particle's final position to first order. The first-order displacement ξ induced by b vanishes initially because the starting positions are fixed. We impose $\xi(T) = 0$ to select $b(0)$.

Fix $0 < T \le 1$ and $I = [0, T]$. Let the parent u be a smooth divergence-free field on $I \times \mathbb{T}^3$, not assumed to solve any equation, and let $X(t, a)$, $X(0, a) = a$, be its volume-preserving flow. Set

$$M = (\nabla u) \circ X, \quad H^\# = -(M_t + M^2), \quad K = (D_a X)^T D_a X, \quad \Omega = (D_a X)^T (M^T - M) D_a X. \tag{4.1}$$

Time derivatives keep a fixed. Choose $D_- \ge \max\{1, \sup_{I,a} |(D_a X)^{-1}|\}$. The flow equation gives

$$\partial_t D_a X = M D_a X, \qquad K_t - \Omega = 2 (D_a X)^T \partial_t D_a X. \tag{4.2}$$

Let p be a smooth function and write $R = u_t + u \cdot \nabla u + \nabla p$. The gradient of this identity is $(\partial_t + u \cdot \nabla)\nabla u + (\nabla u)^2 + D^2 p = \nabla R$, with $(\nabla u)_{ij} = \partial_j u_i$. Along trajectories the material derivative becomes ∂_t, so

$$H^{\#} = (D^2 p - \nabla R) \circ X. \tag{4.3}$$

For a solution of the Euler equations, $H^{\#}$ is the pressure Hessian along the flow: it is what prevents the velocity gradient from evolving by the Riccati equation $M_t = -M^2$, and it is symmetric; for a linear flow Dx this is the condition that $D' + D^2$ be symmetric in Subsection 2.1. In general $H^{\#}$ need not be symmetric. The matrix $M^T - M$ is the vorticity of u along the flow: $(M^T - M)y = -(\operatorname{curl} u \circ X) \times y$. Thus Ω is the vorticity pulled back to labels. When $H^{\#} \leq \mathrm{Id}$ as quadratic forms, the displacement energy is coercive and the inverse in Appendix A gives a unique correction with a Hölder bound.

4.1. **Displacement and the two end conditions.** Here $H^1_0(I;\mathbb{R}^3)$ is the space of $z \in H^1(I;\mathbb{R}^3)$ with $z(0) = z(T) = 0$, the closure of $C^\infty_c((0,T);\mathbb{R}^3)$ in H^1. Fix a label a, let $z \in H^1_0(I;\mathbb{R}^3)$, and put $\xi = D_a X(\cdot, a)z$. Since $\partial_t D_a X = M D_a X$,

$$\xi_t = (\partial_t D_a X)z + D_a X z_t = M\xi + D_a X z_t, \qquad z_t = (D_a X)^{-1}(\xi_t - M\xi). \tag{4.4}$$

By the first variation formula (1.5), the displacement induced by b satisfies $\xi_t = M\xi + b \circ X$. Thus (4.4) identifies $D_a X z_t = b \circ X$. Since $D_a X$ is invertible, the prescribed final displacement $\xi(T) = 0$ is equivalent to $z(T) = 0$.

Cauchy–Schwarz in $z(t) = -\int_t^T z_t(s)\,\mathrm{d}s$ gives

$$|z(t)|^2 \leq (T-t)\int_I |z_t|^2\,\mathrm{d}s, \qquad \int_I |z|^2\,\mathrm{d}t \leq \frac{T^2}{2}\int_I |z_t|^2\,\mathrm{d}t. \tag{4.5}$$

Since $D_a X$ is smooth, $\xi \in H^1_0(I;\mathbb{R}^3)$ as well, so (4.5) also holds for ξ; in particular $\|\xi\|_{L^2(I)} \leq \|\xi_t\|_{L^2(I)}$, since $T \leq 1$. By (4.4) and $|(D_a X)^{-1}| \leq D_-$,

$$\|z_t\|_{L^2(I)} \leq D_-\big(\|\xi_t\|_{L^2(I)} + \|M\|_\infty \|\xi\|_{L^2(I)}\big) \leq D_-(1 + \|M\|_\infty)\|\xi_t\|_{L^2(I)}.$$

Squaring gives

$$\int_I |z_t|^2\,\mathrm{d}t \leq D_-^2(1 + \|M\|_\infty)^2 \int_I |\xi_t|^2\,\mathrm{d}t. \tag{4.6}$$

4.2. **Coercivity.** The inverse in Appendix A requires the label form $\int_I (Kz_t \cdot z_t + \Omega z_t \cdot z)\,\mathrm{d}t$ to control $\int_I |z_t|^2\,\mathrm{d}t$. In the displacement variable ξ, the integrand contains $H^{\#}$ and a total derivative that integrates to zero because ξ vanishes at both ends. We can then use the upper bound on $H^{\#}$ and (4.6) to obtain the required estimate.

Substitute $D_a X z_t = \xi_t - M\xi$ and $D_a X z = \xi$ in the two terms. First, $Kz_t \cdot z_t = |D_a X z_t|^2 = |\xi_t|^2 - 2\xi_t \cdot M\xi + |M\xi|^2$. Second, $\Omega z_t \cdot z = (M^T - M)(\xi_t - M\xi)\cdot\xi = \xi_t \cdot M\xi - \xi_t \cdot M^T\xi - |M\xi|^2 + M^2\xi\cdot\xi$. Adding, and using $\partial_t(\xi \cdot M\xi) = \xi_t \cdot M\xi + \xi_t \cdot M^T\xi + M_t\xi \cdot \xi$ and $M_t + M^2 = -H^{\#}$, we obtain

$$Kz_t \cdot z_t + \Omega z_t \cdot z = |\xi_t|^2 - H^{\#}\xi\cdot\xi - \partial_t(\xi \cdot M\xi).$$

Since ξ vanishes at both ends, integration and density give

$$\int_I (Kz_t \cdot z_t + \Omega z_t \cdot z)\,\mathrm{d}t = \int_I (|\xi_t|^2 - H^{\#}\xi\cdot\xi)\,\mathrm{d}t. \tag{4.7}$$

Lemma 4.1. *Assume $H^{\#} \leq \mathrm{Id}$ on $I \times \mathbb{T}^3$ as quadratic forms. Then for every $a \in \mathbb{T}^3$, $z \in H^1_0(I;\mathbb{R}^3)$ and $\xi = D_a X(\cdot, a)z$, one has $\int_I H^{\#}\xi\cdot\xi\,\mathrm{d}t \leq \frac{1}{2}\int_I |\xi_t|^2\,\mathrm{d}t$. Hence*

$$\int_I (Kz_t \cdot z_t + \Omega z_t \cdot z)\,\mathrm{d}t \geq \frac{1}{2}\int_I |\xi_t|^2\,\mathrm{d}t. \tag{4.8}$$

Proof. The assumption $H^{\#} \le \mathrm{Id}$, the second inequality of (4.5) for ξ and $T \le 1$ give

$$\int_I H^{\#}\xi \cdot \xi \,\mathrm{d}t \le \int_I |\xi|^2 \,\mathrm{d}t \le \frac{T^2}{2}\int_I |\xi_t|^2 \,\mathrm{d}t \le \frac{1}{2}\int_I |\xi_t|^2 \,\mathrm{d}t.$$

Equation (4.7) gives (4.8). □

4.3. **The inverse and its pressure.** Given a source $L(t,a)$, we solve the system

$$K(t)v(t) - \int_0^t (\Omega v)(s)\,\mathrm{d}s + L(t) = c + \nabla_a \phi, \qquad v = z_t, \quad z(0) = z(T) = 0, \quad \mathrm{div}_a\, z = 0, \tag{4.9}$$

for the displacement z, a time-independent divergence-free field c and a label pressure gradient $\nabla_a\phi$. For the force sources of Theorem 4.2, (4.9) is the linearized momentum equation (4.11) integrated once in time. For these sources $L(0) = 0$. Since $K(0) = \mathrm{Id}$, the equation at time zero reads $v(0) = c + \nabla_a\phi(0)$. Projecting onto divergence-free fields gives $c = v(0)$. Since $b \circ X = D_aXv$ and $X(0,a) = a$, the constant c is the initial correction $b(0)$.

Theorem 4.2. *Assume $H^{\#} \le \mathrm{Id}$ on $I \times \mathbb{T}^3$ as quadratic forms. For every $L \in C(I; C^{0,1/2})$ the system (4.9) has a unique solution with $z \in C^1(I; C^{0,1/2})$, $c \in C^{0,1/2}$ divergence free and $\nabla_a\phi \in C(I; C^{0,1/2})$. There are absolute C, p such that, with*

$$A = C\big(D_- + \|M\|_\infty + \|K\|_{C_tC^{1,1/2}} + \|\Omega\|_{C_tC^{1,1/2}}\big)^p, \tag{4.10}$$

one has $\|v\|_{C_tC^{0,1/2}} \le A\,\|L\|_{C_tC^{0,1/2}}$.

Let f be a smooth field on $I \times \mathbb{T}^3$, and let $L = -\int_0^t g(s)\,\mathrm{d}s$ with $g = (D_aX)^T(f \circ X)$. Then z is smooth. Put $B = D_aXv$ and $\mathcal{D} = 2(D_aX)^T\partial_t D_aX$. There is a smooth zero-mean label pressure q such that

$$Kv_t + \nabla_a \mathsf{q} = g - \mathcal{D}v. \tag{4.11}$$

Thus $b = B \circ X^{-1}$ and $q = \mathsf{q} \circ X^{-1}$ are smooth and solve

$$\mathrm{div}\, b = 0, \qquad b_t + u \cdot \nabla b + b \cdot \nabla u + \nabla q = f. \tag{4.12}$$

The endpoint conditions select the initial velocity $b(0)$.

The number A also bounds, uniformly in t, the operator norm on $C^{0,1/2}$ of the fixed-time inverse $(\mathbb{P}K(t))^{-1}\mathbb{P}$, where $(\mathbb{P}K(t))^{-1}$ inverts the map $v \mapsto \mathbb{P}(K(t)v)$ on the divergence-free fields of $C^{0,1/2}$.

Proof. Rescale time by $\tau = t/T$ and put $Z(\tau) = z(t)/T$, so $Z_\tau = v$. Theorem A.1 applies with coefficients $K, T\Omega$, ellipticity D_-^{-2} and source L. Its coercivity constant is $1/\big(2D_-^2(1 + \|M\|_\infty)^2\big)$, by Lemma 4.1 and (4.6). It gives the unique solution. Since $D_- \ge 1$ and $T \le 1$, the quantity Q of Appendix A is at most an absolute constant times a fixed power of the sum in parentheses in (4.10), so the estimate (A.3) gives the asserted bound. For a smooth f the source L is smooth, and so is the solution, by the last assertion of Theorem A.1. Differentiating (4.9), using $K_t - \Omega = \mathcal{D}$ from (4.2), and setting $\mathsf{q} = -\phi_t$ gives (4.11). Then $B_t + MB = D_aXv_t + 2(\partial_t D_aX)v$ and the Piola identity $\mathrm{div}_x(B \circ X^{-1}) = (\mathrm{div}_a\, v) \circ X^{-1}$ give (4.12). Finally, by Lemma A.2 with $H = \mathbb{R}^3$, $\mathcal{R} = \mathbb{P}$ and $B = K(t)$, whose coercivity is D_-^{-2}, the fixed-time inverse is bounded by a fixed power of $2 + D_-^2 + \|K(t)\|_{C^1}$. Since $D_- \ge 1$, this number is also at most an absolute constant times a fixed power of the sum in parentheses in (4.10), so A bounds the fixed-time inverse once C and p are enlarged. □

5. From a localized wave to an approximate solution

We correct the localized wave of Section 3 to obtain the parent velocity for the next wave. The corrections must preserve the leading wave's rank-one contribution to the velocity gradient at the origin, where the next wave is amplified. The new parent must also have controlled flow deformation and satisfy the pressure and residual bound (5.2) for the next endpoint problem.

At each level of the expansion we first remove the phase average of the error using Theorem 4.2. The first mean correction also cancels the residual of the parent. Transport of the leading wave by the new mean then enters the equation for the oscillatory correction. The residual of the new parent has three sources. The depth controls the remainder of the expansion. The wave is small compared with its peak while it is switched on, so its gain controls the error from the time cutoff. The final Fourier cutoff contributes the error estimated in (5.56). Section 7 chooses the parameters to make all three errors small on the scale of the next wave.

5.1. **A step over a truncated parent.**

Throughout this section the parent is the truncated velocity to which we add the new wave. We prescribe the wave vector and seed at the origin and extend the seed to the labels in the envelope. Section 6 verifies the two assumptions (5.5) on the central pair for the first wave and for each later wave.

We use the Fourier modes $e^{2\pi i\xi\cdot x}$ for $\xi\in\mathbb{Z}^3$. A field is *supported on frequencies at most* Λ if its Fourier coefficients vanish for $|\xi|>\Lambda$. Let the parent U be a smooth odd divergence-free field on $[0,T]\times\mathbb{T}^3$, $T\le 1$, and use its flow X, the matrix M and the operators $\mathbf{d}$, $\widetilde{\mathbf{d}}$ of Subsection 3.1. Let $\mathsf{D}\ge 2$ be a number such that U is supported on frequencies at most $2\mathsf{D}$ and

$$\|\nabla U\|_\infty\le \mathsf{D},\qquad \sup|(D_aX)^{\pm1}|\le \mathsf{D}. \tag{5.1}$$

Since U is odd, the flow fixes the origin, and D_aX and M are even in a. Let p_U be the function of zero mean with $\Delta p_U=-\operatorname{tr}((\nabla U)^2)$, and let $R_U=U_t+U\cdot\nabla U+\nabla p_U$ be the residual of the parent; no other equation is assumed. Thus $R_U=U_t+\mathbb{P}\operatorname{div}(U\otimes U)$ is odd, divergence free and supported on frequencies at most $4\mathsf{D}$. To apply the endpoint inverse of Theorem 4.2 to the mean correction over U, assume that

$$D^2p_U-\nabla R_U\le \mathrm{Id}\qquad\text{on } [0,T]\times\mathbb{T}^3 \tag{5.2}$$

as quadratic forms. By (4.3) this is the hypothesis $H^{\#}\le\mathrm{Id}$ of that theorem.

The wave is that of Subsection 3.1 over U, with its constant vector $k_*\neq 0$ and transported field $\kappa=(D_aX)^{-T}k_*$, its activation time $t_0\in[0,T)$ and its cutoff in time $\widetilde{\chi}$. Let $\lambda\ge 1$ be the frequency of the wave: its wave vector at the origin has length $|\kappa(t_0,0)|=\lambda$ at the activation time. Since $k_*=(D_aX)^T\kappa$ at every time and label, the bound (5.1) for the deformation gives

$$\mathsf{D}^{-1}\lambda\le|k_*|\le\mathsf{D}\lambda,\qquad \mathsf{D}^{-2}\lambda\le|\kappa|\le\mathsf{D}^2\lambda. \tag{5.3}$$

The profile is $F_0=F_h$ of Lemma 3.1, with $h=\lambda^{-1/2}$. At the origin the seed $\mathbf{a}(t,0)$, $t_0\le t\le T$, is a solution of the seed equation (3.2) at the label $a=0$, perpendicular to $\kappa(t,0)$ and normalized by $\max_{t_0\le t\le T}|\mathbf{a}(t,0)|=\lambda^{-3/4}$. It does not vanish, because it is a nonzero solution of a linear ordinary differential equation. Put

$$\gamma(t)=\lambda^{3/4}|\mathbf{a}(t,0)|,\qquad \nu(t,0)=\Big(\frac{\kappa}{|\kappa|}\cdot M\frac{\mathbf{a}}{|\mathbf{a}|}\Big)(t,0)\qquad (t_0\le t\le T).$$

As in Lemma 2.2, $\nu(t,0)$ is the first coupling of the pair $(\kappa,\mathbf{a})$ at the origin over the parent, and $|\mathbf{a}(t,0)||\kappa(t,0)|$ is its strength. At the other labels the seed is the solution of (3.2) on $[t_0,T]$ with

$$\mathbf{a}(t_0,a)=\big(\mathrm{Id}-P_{\kappa(t_0,a)}\big)\mathbf{a}(t_0,0),\qquad P_\kappa=\frac{\kappa\otimes\kappa}{|\kappa|^2}, \tag{5.4}$$

the projection of the vector at the origin onto the plane perpendicular to $\kappa(t_0,a)$. It is even in a.

Fix a smooth function χ on $[0,\infty)$ with $0 \le \chi \le 1$, $\chi = 1$ on $[0,1]$ and $\chi = 0$ on $[2,\infty)$, and take the even envelope $g(a) = \chi(|a|/\ell_*)$, equal to one on the ball of radius ℓ_* specified below. The leading oscillation is the field $A_1 = g\mathbf{a}F_h$ of (3.3), and it enters through $V_1 = \widetilde{\chi}A_1$ of (3.7).

At the origin, assume that for $t_0 \le t' \le t \le T$,

$$\frac{|\mathbf{b}(t)|}{\gamma(t)} \le \mathsf{D}\,\frac{|\mathbf{b}(t')|}{\gamma(t')}, \qquad \nu(t,0) \ge \mathsf{D}^{-1}, \tag{5.5}$$

the first for every solution $\mathbf{b}$ of the seed equation at the origin that is perpendicular to $\kappa(\cdot,0)$. The first inequality bounds the growth of every transverse solution by that of the seed, up to the factor D. On a sufficiently small envelope this gives the weighted estimate for the transverse corrections, and the second inequality keeps $\kappa \cdot M\mathbf{a}$ positive there (Lemma 5.3). With this sign, the lower bound $F_h' \ge -Ch$ controls the leading pressure tensor in (5.51) from above. Both inequalities can be imposed before choosing the frequency and the depth of the expansion.

On the envelope, (5.19) bounds the seed by twice its size at the origin. Since $\|F_h\|_\infty \le Ch$, the leading velocity has size at most $C\lambda^{-3/4}h = C\lambda^{-5/4}$. The first mean equation removes both the quadratic wave forcing and R_U, so we require the parent residual to be at most the square of this scale,

$$\|R_U\|_{C_tL^\infty} \le \lambda^{-5/2}. \tag{5.6}$$

The radius of the envelope is a negative power of D with an absolute exponent, $\ell_* \le \mathsf{D}^{-3}$. The exponent is fixed sufficiently large for the finitely many estimates of Subsections 5.2–5.4. In particular the support of g lies in the interior of $[-1/2,1/2]^3$.

Constants and exponents written C are absolute: they do not depend on the parent, the wave, an order of differentiation or the depth of the expansion. A constant written C_n depends only on the order n, on the fixed profiles, and on the constants of the same kind in the hypotheses of the statement in which it appears; $C_{n,p}$ and $C_{m,P}$ depend also on a level p or on the depth P. These constants may change from one estimate to the next, and no bound on their growth in n is used. In the iteration these constants and the powers of D are at most a small power of λ; see (7.8).

5.2. **Coefficient bounds for the parent.** The coefficients of the two linear problems are built from the deformation D_aX of the parent, its velocity gradient M and the wave vector κ. Bernstein's inequality (B.1) for the truncated parent and the differentiated flow equation bound each label derivative of these coefficients at the cost of a fixed power of D. The norm $\|\cdot\|_{[n]}$ of (D.1) weights each label derivative by the radius ℓ_* and each phase derivative by the width h. For a coefficient $c(t,a)$ it is $\sum_{m\le n}\ell_*^m|c|_m$, where $|c|_m$ sums the suprema of its label derivatives of order m. Taking ℓ_* sufficiently small absorbs the cost of each derivative, so the exponent of D in (5.8) is independent of n.

Proposition 5.1. *There is an absolute exponent C such that each coefficient c among*

$$(D_aX)^{\pm1}, \qquad M, \qquad \kappa/\lambda, \qquad \lambda^2/|\kappa|^2, \qquad P_\kappa$$

satisfies (D.6). Let $f(t,x)$ be a field with values in a normed space such that $|f|_m \le C_m'A_f\mathsf{D}^m$ for every $m \ge 0$ and a number A_f. Then

$$|f\circ X|_m \le C_mA_f\mathsf{D}^{Cm} \qquad (m \ge 0). \tag{5.7}$$

Consequently, for every $n \ge 0$,

$$\|c\|_{[n]} \le C_n\mathsf{D}^C, \qquad \|f\circ X\|_{[n]} \le C_nA_f. \tag{5.8}$$

Proof. For $n \ge 1$ let $\upsilon_n = \max_i |X_i|_n$, so $\upsilon_1 \le 3\mathsf{D}$. By (B.1) for ∇U, with values in the continuous functions of time, and (5.1), $|U|_k \le C_k\mathsf{D}^{k-1}\|\nabla U\|_\infty \le C_k\mathsf{D}^k$ for $k \ge 1$. For $|\beta| = n \ge 2$, the chain rule gives

$$\partial_t\partial^\beta X = (\nabla U\circ X)\partial^\beta X + N_\beta,$$

where N_β is a sum, with coefficients that depend only on β, of the terms

$$\big((\partial^{\beta'}U)\circ X\big)\prod_{j=1}^{k}\partial^{\beta_j}X_{l_j},\qquad 2\le k=|\beta'|\le n,\quad \beta_1+\cdots+\beta_k=\beta,\quad |\beta_j|\ge 1.$$

Since $\partial_t D_a X=(\nabla U\circ X)D_aX$ and $\partial^\beta X(0)=0$,

$$\partial^\beta X(t)=\int_0^t D_aX(t)(D_aX(\sigma))^{-1}N_\beta(\sigma)\,\mathrm{d}\sigma.$$

By (5.1) the propagator has norm at most D^2, and $T\le 1$. Hence, for $n\ge 2$,

$$\upsilon_n\le C_n\mathsf{D}^2\sum_{k=2}^{n}\mathsf{D}^k\sum_{n_1+\cdots+n_k=n}\upsilon_{n_1}\cdots\upsilon_{n_k},\tag{5.9}$$

with every $n_i\ge 1$. Induction on n gives $\upsilon_n\le C_n\mathsf{D}^{5n-4}$: by the bounds at the orders $n_i<n$, a term of (5.9) with k factors is at most C_n times

$$\mathsf{D}^{2+k}\,\mathsf{D}^{5n-4k}=\mathsf{D}^{5n-4-3(k-2)}\le\mathsf{D}^{5n-4}.$$

Thus $|D_aX|_m\le C_m\mathsf{D}^{5m+1}$ for every $m\ge 0$. For f as in the statement, the chain rule bounds $|f\circ X|_n$, $n\ge 1$, by C_n times the terms $A_f\mathsf{D}^k\mathsf{D}^{5n-4k}$ with $1\le k\le n$, which proves (5.7). By (B.1), $f=\nabla U$ satisfies the hypothesis with $A_f=\|\nabla U\|_\infty\le\mathsf{D}$, which bounds M.

The product and inverse rules of Lemma D.2(a),(b) now apply successively to $(D_aX)^{-1}$, $\kappa/\lambda=(D_aX)^{-T}k_*/\lambda$, $|\kappa/\lambda|^{-2}$ and P_κ. Indeed, (5.1) bounds the inverse deformation, and (5.3) gives $|k_*|/\lambda\le\mathsf{D}$ and $|\kappa|/\lambda\ge\mathsf{D}^{-2}$. Finally, ℓ_* is at most the power D^{-C} that occurs here. So part (c) gives the first bound of (5.8), and $\ell_*^m|f\circ X|_m\le C_mA_f(\mathsf{D}^C\ell_*)^m\le C_mA_f$ gives the second. □

In particular the residual of the parent, which is supported on frequencies at most $4\mathsf{D}$, satisfies the hypothesis of (5.7) with $A_f=\|R_U\|_{C_tL^\infty}$, by (B.1). With (5.6) this gives, for every $n\ge 0$,

$$\|R_U\circ X\|_{[n]}\le C_n\lambda^{-5/2}.\tag{5.10}$$

5.3. **Finite-order estimates for the mean.** The phase-independent corrections of the expansion solve the linearized Euler equations over the parent, with zero displacement at both ends of $[0,T]$. For a smooth phase-independent source $\bar f(t,a)$, Theorem 4.2 over U gives $(B,\bar q)$ with

$$B_t+MB+\mathbf{d}\bar q=\bar f,\qquad \mathbf{d}\cdot B=0,\qquad z(0)=z(T)=0,\quad B=D_aXz_t.\tag{5.11}$$

In the notation of that theorem, $\bar f=f\circ X$, $g=(D_aX)^T\bar f$, $v=z_t$, and $\bar q$ is the pressure q of (4.11). We call (5.11) the *mean equation*. The terminal condition selects the initial velocity $B(0)$, so the mean can already be present before activation. We estimate it on all of $[0,T]$ in the unweighted label norms. The inverse preserves the derivative order in the scaled Hölder norms. Passing to the supremum norms by (D.2) costs one derivative of the source. At each level of the expansion the mean source is formed from terms already estimated at all orders.

Proposition 5.2. *There is an absolute exponent C such that, for every $n\ge 0$, every solution of the mean equation (5.11) satisfies*

$$\|B\|_{[n]}\le C_n\mathsf{D}^C\left\|\bar f\right\|_{[n+1]}.\tag{5.12}$$

Proof. By Proposition 5.1 and Lemma D.2(a), the coefficients K and Ω of (4.1), which enter (4.9), satisfy (D.6). Theorem 4.2 applies over U with $D_-=\mathsf{D}$, by (5.1). Its constant A of (4.10) contains D_-, $\|M\|_\infty\le\mathsf{D}$ and the norms of K and Ω in $C_tC^{1,1/2}$, so it is at most D^C. Since ℓ_* is a sufficiently large negative power of D, the first bound of (D.2) and Lemma D.2(c) give

$$\ell_*^m\big(|K|_{m,1/2}+|\Omega|_{m,1/2}\big)\le C_m\mathsf{D}^C\ell_*^{1/2}\le C_m/A\qquad(m\ge 1).\tag{5.13}$$

Label derivatives preserve the endpoint and divergence constraints, the time independence of the divergence-free field, and the class of label gradients in (4.9). Theorem 4.2 bounds the map $L \mapsto v$ by A in $C_t C^{0,1/2}$. The term Kv enters directly, and Ωv enters through time integration, an operator of norm at most $T \le 1$. Thus (5.13) gives the coefficient bounds for Lemma D.3 with $C_{\mathrm{low}} = A$. Applying its Hölder estimate to the source $L = -\int_0^t g(t')\,\mathrm{d}t'$ of (4.9) gives

$$\|v\|_{[n],1/2} \le C_n \mathsf{D}^C \, \|g\|_{[n],1/2}\,. \tag{5.14}$$

By (D.2) and (5.8), $\|D_a X\|_{[n],1/2} \le C_n \ell_*^{-1/2} \mathsf{D}^C \le C_n \mathsf{D}^{C'}$, with C' absolute. Using $B = D_a X v$ and $g = (D_a X)^T \bar{f}$ in (5.14), the product estimate (D.4) gives $\|B\|_{[n],1/2} \le C_n \mathsf{D}^C \|\bar{f}\|_{[n],1/2}$. The comparison (D.2) now gives (5.12). □

5.4. **The transverse equation.** Each oscillatory correction solves a forced seed equation forward from zero initial data at each label and phase. Its forcing vanishes before activation, so the correction does too. From activation onward we measure its size relative to the amplitude of the seed at the origin, using the weight γ.

We call

$$V_t + MV + \kappa \partial_s \mathsf{p} = f, \qquad \kappa \cdot V = 0, \qquad \langle V \rangle_s = \langle f \rangle_s = 0, \tag{5.15}$$

the *transverse equation*, because its unknown V is perpendicular to the wave vector κ. The pressure term keeps V perpendicular to κ: differentiating $\kappa \cdot V = 0$ and using $\kappa_t = -M^T \kappa$ in (5.15) gives

$$|\kappa|^2 \partial_s \mathsf{p} = \kappa \cdot f - 2\kappa \cdot MV. \tag{5.16}$$

The transverse equation is therefore the linear ordinary differential equation

$$V_t = -(\mathrm{Id} - 2P_\kappa) MV + (\mathrm{Id} - P_\kappa) f \tag{5.17}$$

at each fixed (a, s); for $f = 0$ it is the seed equation (3.2). For every source we take its solution with $V(0, a, s) = 0$, and p is the zero-mean primitive in s of (5.16). For smooth f the solution is unique and smooth, and V satisfies the two constraints in (5.15). Both V and p vanish at every label where f vanishes for all (t, s). If f vanishes for $t < t_0$, then $V = 0$ for $t \le t_0$.

The plane $\kappa^\perp$ varies with the label. To compare the equation on the envelope with the one at the origin and to differentiate it in the labels, we express V in coordinates in a fixed $\mathbb{R}^2$. Choose a constant 3×2 matrix with orthonormal columns spanning $k_*^\perp$, let $A_\perp$ be $D_a X$ times this matrix, and let $A_\perp^\dagger = (A_\perp^T A_\perp)^{-1} A_\perp^T$. The columns of $A_\perp$ span $\kappa^\perp$, since $\kappa = (D_a X)^{-T} k_*$, and $A_\perp^\dagger$ takes a vector to its coordinates in those columns after orthogonal projection onto their span: $A_\perp^\dagger A_\perp = \mathrm{Id}_{\mathbb{R}^2}$ and $A_\perp A_\perp^\dagger = \mathrm{Id} - P_\kappa$. Write $V = A_\perp v$ for a transverse velocity. Since $\partial_t A_\perp = M A_\perp$, the equation (5.17) becomes

$$v_t + \mathcal{M}_\perp v = \psi, \qquad \mathcal{M}_\perp = 2A_\perp^\dagger M A_\perp, \qquad \psi = A_\perp^\dagger f. \tag{5.18}$$

Lemma 5.3. *For $t_0 \le t \le T$ and $|a| \le 2\ell_*$, the seed satisfies*

$$|\mathbf{a}(t, a)| \le 2|\mathbf{a}(t, 0)|, \qquad (\kappa \cdot M\mathbf{a})(t, a) > 0. \tag{5.19}$$

There is an absolute exponent C such that the following holds at every order $n \ge 0$.

(i) Let V and f be smooth fields on $[t_0, T]$ that satisfy (5.17), with V perpendicular to κ, and suppose that $V(t_0)$ and f vanish at the labels outside $\operatorname{supp} g$. *Then*

$$\|V\|_{[n],\gamma} \le C_n \mathsf{D}^C \big(\gamma(t_0)^{-1} \|V(t_0)\|_{[n]} + \|f\|_{[n],\gamma} \big). \tag{5.20}$$

(ii) Let f be smooth on $[0, T]$ with zero phase mean, and suppose that f vanishes for $t < t_0$ and at the labels outside $\operatorname{supp} g$. *The solution of (5.15) with $V(0) = 0$ and its pressure satisfy*

$$\|V\|_{[n],\gamma} + \lambda \|\mathsf{p}\|_{[n],\gamma} \le C_n \mathsf{D}^C \|f\|_{[n],\gamma}. \tag{5.21}$$

(iii) On $[t_0, T]$ the leading oscillation and its pressure satisfy

$$\|A_1\|_{[n],\gamma} + \lambda \|\mathsf{p}[A_1]\|_{[n],\gamma} \leq C_n \mathsf{D}^C \lambda^{-5/4}. \tag{5.22}$$

Proof. By (5.1), $\mathsf{D}^{-1}|w| \leq |A_\perp w| \leq \mathsf{D}|w|$ for $w \in \mathbb{R}^2$. Hence $|A_\perp| \leq \mathsf{D}$ and $|A_\perp^\dagger| \leq \mathsf{D}$, and the matrix $K_\perp = A_\perp^T A_\perp$ satisfies $\sup |K_\perp^{-1}| \leq \mathsf{D}^2$. By Proposition 5.1 and Lemma D.2(a),(b), the coefficients $A_\perp$, $A_\perp^\dagger = K_\perp^{-1} A_\perp^T$ and $\mathcal{M}_\perp$ satisfy (D.6).

In the coordinates $V = A_\perp v$, the seed equation at the label a is $v_t + \mathcal{M}_\perp(t,a)v = 0$. With the bounds for $A_\perp$ and $A_\perp^\dagger$, the first condition in (5.5) gives $|v(t)|/\gamma(t) \leq \mathsf{D}^3 |v(t')|/\gamma(t')$ for its solutions at $a = 0$. By the bound for the first label derivatives of $\mathcal{M}_\perp$ and the choice of ℓ_*,

$$\sup_{t_0 \leq t \leq T} |\mathcal{M}_\perp(t,a) - \mathcal{M}_\perp(t,0)| \leq C\mathsf{D}^C |a| \leq \mathsf{D}^{-3} \qquad (|a| \leq 2\ell_*).$$

Let v solve the equation at such a label a on $[t', T]$. It solves the equation at the origin with the source $-(\mathcal{M}_\perp(\cdot, a) - \mathcal{M}_\perp(\cdot, 0))v$, so variation of constants gives

$$\frac{|v(t)|}{\gamma(t)} \leq \mathsf{D}^3 \frac{|v(t')|}{\gamma(t')} + \int_{t'}^t \frac{|v(t'')|}{\gamma(t'')} \, \mathrm{d}t''.$$

The weights cancel because the bound at the origin is a ratio of values of γ. Since $t - t' \leq T \leq 1$, Gronwall's inequality gives $|v(t)|/\gamma(t) \leq e\mathsf{D}^3 |v(t')|/\gamma(t')$. For a solution of (5.18) on $[t_0, T]$, variation of constants therefore gives, at each phase and each label with $|a| \leq 2\ell_*$,

$$\frac{|v(t,a)|}{\gamma(t)} \leq e\mathsf{D}^3 \left(\frac{|v(t_0,a)|}{\gamma(t_0)} + \int_{t_0}^t \frac{|\psi(t',a)|}{\gamma(t')} \, dt' \right). \tag{5.23}$$

To compare the seed at a with its value at the origin, write $\mathbf{a} = A_\perp v_a$, with $v_a = A_\perp^\dagger \mathbf{a}$ at a. The difference $v_a - v_0$ solves the equation at the origin with the source $-(\mathcal{M}_\perp(\cdot, a) - \mathcal{M}_\perp(\cdot, 0))v_a$. Its value at t_0 is $(A_\perp^\dagger(t_0, a) - A_\perp^\dagger(t_0, 0))\mathbf{a}(t_0, 0)$, by (5.4) and $A_\perp^\dagger P_\kappa = 0$, and $|\mathbf{a}(t_0, 0)| = \gamma(t_0)\lambda^{-3/4}$. With the bound just proved for v_a, variation of constants gives $|v_a(t) - v_0(t)| \leq C\mathsf{D}^C |a| \gamma(t) \lambda^{-3/4}$. Since $\gamma(t)\lambda^{-3/4} = |\mathbf{a}(t, 0)|$, it follows that

$$|\mathbf{a}(t,a) - \mathbf{a}(t,0)| \leq C\mathsf{D}^C |a| \, |\mathbf{a}(t,0)|, \qquad t_0 \leq t \leq T.$$

The label derivatives also give $|\kappa(t,a) - \kappa(t,0)| \leq C\mathsf{D}^C |a| |\kappa(t,0)|$ and $|M(t,a) - M(t,0)| \leq C\mathsf{D}^C |a|$. Since $|M| \leq \mathsf{D}$, the three bounds give

$$\big|(\kappa \cdot M\mathbf{a})(t,a) - (\kappa \cdot M\mathbf{a})(t,0)\big| \leq C\mathsf{D}^C |a| \, |\kappa(t,0)| \, |\mathbf{a}(t,0)|,$$

while $(\kappa \cdot M\mathbf{a})(t,0) = \nu(t,0)|\kappa(t,0)||\mathbf{a}(t,0)| \geq \mathsf{D}^{-1}|\kappa(t,0)||\mathbf{a}(t,0)|$, by the second condition in (5.5). None of these exponents depends on the exponent of ℓ_*, so the choice of ℓ_* proves (5.19).

Let V and f be as in (i). The solution vanishes at the labels outside $\operatorname{supp} g$, where its initial value and source vanish, and $\operatorname{supp} g$ lies in $\{|a| \leq 2\ell_*\}$. Taking the supremum in (5.23) gives, because $t - t_0 \leq T \leq 1$,

$$\|v\|_{[0],\gamma} \leq e\mathsf{D}^3 \big(\gamma(t_0)^{-1} \|v(t_0)\|_{[0]} + \|\psi\|_{[0],\gamma}\big).$$

Since $\operatorname{supp} g$ is closed, the label and phase derivatives of v and ψ also vanish outside it. So this is the zero-order estimate of Lemma D.3 on $[t_0, T]$, for the initial value and the source together, with the one coefficient $\mathcal{M}_\perp$, which enters through the identity. By the choice of ℓ_*, condition (D.7) holds, and the lemma gives $\|v\|_{[n],\gamma} \leq C_n \mathsf{D}^3 (\gamma(t_0)^{-1} \|v(t_0)\|_{[n]} + \|\psi\|_{[n],\gamma})$. The estimate divides by γ after differentiation in label and phase. Since $V = A_\perp v$, $v = A_\perp^\dagger V$ and $\psi = A_\perp^\dagger f$, the product estimate (D.4) and Lemma D.2(c) for $A_\perp$ and $A_\perp^\dagger$ give (5.20).

In (ii) the forcing vanishes for $t < t_0$, so $V(t_0) = 0$. Applying (5.20) to (5.17) bounds V. By (5.16), $\lambda\mathsf{p}$ is the phase primitive of a product of the coefficients κ/λ and $\lambda^2/|\kappa|^2$ with $f - 2MV$, so

(D.5), (D.4) and (5.8) give

$$\lambda\|\mathsf{p}\|_{[n],\gamma} \le C_n \mathsf{D}^C \big(\|f\|_{[n],\gamma} + \|V\|_{[n],\gamma}\big), \tag{5.24}$$

which proves (5.21).

For (iii), the time independence of the envelope implies that $g\mathbf{a}$ solves the seed equation on $[t_0, T]$ at every label. It is perpendicular to κ and vanishes outside $\operatorname{supp} g$. By (5.4), its value at the time t_0 is $g(\mathrm{Id} - P_\kappa)\mathbf{a}(t_0, 0)$. The chain rule gives $|g|_m \le C_m \ell_*^{-m}$, so $\|g\|_{[n]} \le C_n$. With (5.8) for P_κ and (D.4), this gives $\|g\mathbf{a}(t_0,\cdot)\|_{[n]} \le C_n \mathsf{D}^C |\mathbf{a}(t_0, 0)| = C_n \mathsf{D}^C \gamma(t_0)\lambda^{-3/4}$. Hence (5.20) with $f = 0$ gives $\|g\mathbf{a}\|_{[n],\gamma} \le C_n \mathsf{D}^C \lambda^{-3/4}$. For a function of the phase alone, $\|F\|_{[n]} = \sum_{j\le n} h^j \|\partial_s^j F\|_\infty$, so Lemma 3.1 gives $\|F_h\|_{[n]} \le C_n h$. By (D.4), $\|A_1\|_{[n],\gamma} \le C_n \mathsf{D}^C h\lambda^{-3/4} = C_n \mathsf{D}^C \lambda^{-5/4}$. By (3.4), on $[t_0, T]$ the leading oscillation solves (5.15) with $f = 0$ and $\mathsf{p} = \mathsf{p}[A_1]$, so (5.24) gives (5.22). □

5.5. **The finite expansion.** At depth P we seek a perturbation

$$W = \sum_{p=1}^{P} Z_p + C_{P+1}, \qquad Z_p = Y_p + B_p, \qquad Y_p = V_p + C_p, \qquad C_1 = 0,$$

where V_p is perpendicular to κ and B_p is independent of phase. The divergence correction $C_{p+1} = C[V_p]$ makes $V_p + C_{p+1} = \widetilde{\mathbf{d}} \times Q[V_p]$ divergence free by (3.6), which accounts for the last term C_{P+1}. We call p the *level* of Z_p and of Y_p, V_p, C_p and B_p. The factor $|\kappa|^{-1}$ in the vector potential gives the divergence correction C_p of V_{p-1} the bound of a term of level p, so we group it with V_p.

5.5.1. *The equations at each level.* Start from V_1, p_1, C_2 in (3.7) and put $B_1 = 0$, so that $Y_1 = Z_1 = V_1$. Set aside the activation error $R_{\widetilde{\chi}}$ of (3.8). The terms left by the leading wave at level two are

$$H_2 = \widetilde{\chi}\big[(\partial_t + M)C[A_1] + \mathbf{d}\mathsf{p}[A_1]\big] + (V_1 \cdot \mathbf{d})V_1 + (C_2 \cdot \kappa)\partial_s V_1. \tag{5.25}$$

The linear term has zero phase average because $C[A_1]$ and $\mathsf{p}[A_1]$ do. In the phase average of the last term, (3.6) and integration by parts over a period give

$$\langle (C_2 \cdot \kappa)\partial_s V_1\rangle_s = -\langle \partial_s (C_2 \cdot \kappa) V_1\rangle_s = \langle (\mathbf{d} \cdot V_1)V_1\rangle_s.$$

Thus both quadratic terms in $\langle H_2\rangle_s$ have a single label derivative and two factors of V_1. Each factor has size $C_n \mathsf{D}^C \lambda^{-5/4}$ by (5.22), and the label derivative costs a power of D by (D.3). Hence $\|\langle H_2\rangle_s\|_{[n]} \le C_n \mathsf{D}^C \lambda^{-5/2}$. The parent residual $R_U \circ X$ satisfies a bound of this same order by (5.10), so it can be included in the first mean equation.

In general, a product $(Z_i \cdot \mathbf{d})Z_j$ has level $i + j$. The fast derivative costs one level, so $(Z_i \cdot \kappa)\partial_s Y_j$ has level $i + j - 1$. Since $V_i \cdot \kappa = 0$, only C_i and B_i can differentiate the phase. We collect the terms involving $(B_i \cdot \kappa)\partial_s Y_j$ after the mean solve. The other terms at level $p \ge 3$ give

$$\begin{aligned} H_p = {} & (\partial_t + M)C_p + \mathbf{d}\mathsf{p}_{p-1} \\ & + \sum_{i+j=p} (Z_i \cdot \mathbf{d})Z_j + \sum_{\substack{i+j=p+1 \\ i\ge 2}} (C_i \cdot \kappa)\partial_s Y_j, \end{aligned} \tag{5.26}$$

where all summation indices are positive integers. For every $p \ge 2$ put

$$f_p = -(H_p - \langle H_p\rangle_s), \qquad \bar{f}_p = -\langle H_p\rangle_s \quad (p \ge 3), \qquad \bar{f}_2 = -\langle H_2\rangle_s - R_U \circ X. \tag{5.27}$$

Every profile in H_p is determined by the fields V_i, p_i and B_i with $i < p$. Apply (5.11) to $\bar{f}_p$, obtaining $B_p, \bar{q}_p$. Then put

$$f_p^S = -\sum_{i=2}^{p} (B_i \cdot \kappa)\partial_s Y_{p+1-i}, \tag{5.28}$$

and apply (5.15) to $f_p + f_p^S$, obtaining V_p, p_p. Finally set $Q_p = Q[V_p]$ and $C_{p+1} = C[V_p]$. The term $-(B_p \cdot \kappa)\partial_s V_1$ in (5.28) accounts for the transport of the leading wave by the new mean, so B_p must be found before V_p. Every term in (5.28) has zero phase average since B_i and κ are phase independent.

Since $D_a X$, M and κ are even in a, $R_U \circ X$ is odd in a and $A_1 = g\mathbf{a}F_h$ changes sign under the reflection $(a, s) \mapsto (-a, -s)$, the construction is symmetric under this reflection: every V_p, C_p and B_p changes sign under it, and $W^\natural$ is odd because X is odd.

Proposition 5.4. *For every integer $P \geq 2$, the profiles obtained from (5.11), (5.27) and (5.28) are well defined, and*

$$W = \sum_{p=1}^{P} Z_p + C_{P+1} = \sum_{p=1}^{P}(V_p + C_{p+1} + B_p), \qquad \mathsf{P} = \sum_{p=1}^{P}(\mathsf{p}_p + \bar{q}_p), \quad \bar{q}_1 = 0,$$

satisfy $\widetilde{\mathbf{d}} \cdot W = 0$ and

$$W_t + MW + (W \cdot \widetilde{\mathbf{d}})W + \widetilde{\mathbf{d}}\mathsf{P} + R_U \circ X = R_P + R_{\widetilde{\chi}}. \tag{5.29}$$

Since $\langle V_p \rangle_s = \langle C_p \rangle_s = 0$, the oscillatory part $W_{\mathrm{osc}} = W - \langle W \rangle_s = \sum_{p=1}^{P}(V_p + C_{p+1})$ is supported in $\operatorname{supp} g$*, and so are the sources $f_p + f_p^S$. The fields V_p, C_p, p_p and $f_p + f_p^S$ vanish for $t < t_0$. Set $V_{P+1} = B_{P+1} = 0$, so that $Y_{P+1} = Z_{P+1} = C_{P+1}$. The remainder of the expansion is*

$$\begin{aligned} R_P = {} & (\partial_t + M)C_{P+1} + \mathbf{d}\mathsf{p}_P \\ & + \sum_{\substack{i,j \leq P+1 \\ i+j \geq P+1}} (Z_i \cdot \mathbf{d})Z_j + \sum_{\substack{i,j \leq P+1 \\ i+j \geq P+2}} (Z_i \cdot \kappa)\partial_s Y_j. \end{aligned} \tag{5.30}$$

At the labels outside $\operatorname{supp} g$ *it is independent of the phase. The construction is jointly smooth.*

Proof. By (3.6) and (5.11),

$$\widetilde{\mathbf{d}} \cdot W = \sum_{p=1}^{P} \big[\widetilde{\mathbf{d}} \cdot (\widetilde{\mathbf{d}} \times Q_p) + \mathbf{d} \cdot B_p\big] = 0.$$

With $W = \sum_{i=1}^{P+1} Z_i$, the quadratic term is

$$(W \cdot \widetilde{\mathbf{d}})W = \sum_{i,j}(Z_i \cdot \mathbf{d})Z_j + \sum_{i,j}((C_i + B_i) \cdot \kappa)\partial_s Y_j$$

with $1 \leq i, j \leq P + 1$. For $2 \leq p \leq P$, the terms to be canceled at level p sum to $H_p - f_p^S$, together with $R_U \circ X$ at $p = 2$. The new linear terms

$$(\partial_t + M)(V_p + B_p) + \kappa\partial_s \mathsf{p}_p + \mathbf{d}\bar{q}_p = f_p + f_p^S + \bar{f}_p$$

cancel them, since the right side is $-H_p + f_p^S$ for $p \geq 3$ and $-H_2 + f_2^S - R_U \circ X$ for $p = 2$. For V_1, (3.4) and (3.7) give

$$V_{1,t} + MV_1 + \kappa\partial_s \mathsf{p}_1 = \widetilde{\chi}' A_1, \qquad (\partial_t + M)C_2 + \mathbf{d}\mathsf{p}_1 = \widetilde{\chi}\big[(\partial_t + M)C[A_1] + \mathbf{d}\mathsf{p}[A_1]\big] + \widetilde{\chi}' C[A_1].$$

The two terms with $\widetilde{\chi}'$ sum to $R_{\widetilde{\chi}}$, and the remaining term is canceled in H_2. The surviving products have $i + j > P$ for $\mathbf{d}$ and $i + j - 1 > P$ for $\kappa\partial_s$. Together with the two linear terms left at level $P + 1$, these give (5.30).

The sources at each level are smooth when the preceding profiles are. The transverse equation is a linear ordinary differential equation at fixed (a, s) with smooth coefficients, the mean equation has smooth solutions by Theorem 4.2, and taking the curl preserves smoothness. By induction on the level, the profiles are well defined and jointly smooth.

For the supports, let $\mathcal{O}$ be the open set of pairs (t,a) with $t<t_0$ or $a\notin\operatorname{supp} g$. The fields $V_1=\widetilde{\chi}g\mathbf{a}F_h$, C_2 and p_1 vanish on $\mathcal{O}$, and so does the first term of H_2. If V_i, C_{i+1} and p_i vanish on $\mathcal{O}$ for every $i<p$, then on $\mathcal{O}$ the field H_p contains only the phase-independent products $(B_i\cdot\mathbf{d})B_j$, and $f_p=f_p^S=0$ there. The solution of (5.15) then vanishes on $\mathcal{O}$, by the two properties recorded after (5.17), and so do $C_{p+1}=\mathbf{d}\times Q[V_p]$ and, by (5.16), p_p, the zero-mean primitive of a function that vanishes on $\mathcal{O}$. This proves the assertions on the supports. On $\mathcal{O}$ the remainder (5.30) therefore consists of the products $(B_i\cdot\mathbf{d})B_j$, which are independent of the phase. The means B_p do not vanish before t_0 in general. □

5.5.2. *Bounds at every order.* To bound the linear terms in (5.25) and (5.26), we use the transverse equation to evaluate the time derivative of the curl correction.

Lemma 5.5. *There is an absolute exponent C such that, for every $n\geq 0$ and every field V with $\langle V\rangle_s=0$ and $\kappa\cdot V=0$,*

$$\|C[V]\|_{[n],\gamma}\leq C_n\mathsf{D}^C\lambda^{-1}\|V\|_{[n+1],\gamma}. \tag{5.31}$$

If moreover $V_t+MV+\kappa\partial_s\mathsf{p}=f$ with $\langle f\rangle_s=0$ and $\langle\mathsf{p}\rangle_s=0$, then

$$\|(\partial_t+M)C[V]+\mathbf{d}\mathsf{p}\|_{[n],\gamma}\leq C_n\mathsf{D}^C\lambda^{-1}\big(\|V\|_{[n+1],\gamma}+\|f\|_{[n+1],\gamma}\big). \tag{5.32}$$

For the leading oscillation $V=A_1$ on $[t_0,T]$, the source in this estimate is $f=0$.

Proof. By (3.5), $Q[V]$ is λ^{-1} times the phase primitive of V, multiplied by the coefficients κ/λ and $\lambda^2/|\kappa|^2$, and $C[V]=\mathbf{d}\times Q[V]$. Since ℓ_*^{-1} is a power of D, (D.4), (D.3), (D.5) and (5.8) give (5.31).

For (5.32) use $\kappa_t=-M^T\kappa$, $(|\kappa|^{-2})_t=2(\kappa\cdot M\kappa)|\kappa|^{-4}$, and $[\partial_t,\mathbf{d}_j]=-M_{\ell j}\mathbf{d}_\ell$. Differentiating $Q[V]$ in time and inserting the equation for V_t, in which the term $\kappa\times\kappa\,\partial_s\mathsf{p}$ vanishes, gives

$$\partial_tQ[V]=\partial_s^{-1}\left[\frac{(M^T\kappa)\times V+\kappa\times MV-\kappa\times f}{|\kappa|^2}-2\frac{(\kappa\times V)(\kappa\cdot M\kappa)}{|\kappa|^4}\right]. \tag{5.33}$$

Commuting ∂_t past the curl gives $(\partial_t+M)C[V]=\mathbf{d}\times\partial_tQ[V]+[\partial_t,\mathbf{d}\times]Q[V]+MC[V]$. By (5.33) and the formula for $\partial_s\mathsf{p}$ in (5.16), each term, including $\mathbf{d}\mathsf{p}$, carries a factor λ^{-1} and at most one label derivative. The product and primitive estimates therefore give (5.32) without a time derivative of M, f or a mean. The weight γ is applied after the time differentiation. □

Up to fixed powers of D, the leading oscillation has size $\lambda^{-5/4}$ by (5.22), its divergence correction has size $\lambda^{-9/4}$ by (5.31), and the first mean has size $\lambda^{-5/2}$ by the calculation after (5.25) and (5.12). A label derivative costs a power of D, while a phase derivative in Eulerian advection costs $|\kappa|h^{-1}\leq\mathsf{D}^2\lambda^{3/2}$. The smallest gain in frequency comes from advection of the phase by the divergence correction, which contributes $\lambda^{-9/4}\lambda^{3/2}=\lambda^{-3/4}$. Advection of the phase by the mean gives λ^{-1}, as does taking a divergence correction. Advection in the labels gives $\lambda^{-5/4}$. Allowing $\lambda^{-2/3}$ per level leaves a factor $\lambda^{-1/12}$ to absorb the parent bounds.

We induct on the level at all derivative orders at once. When a source requires a higher derivative of a previously computed profile, that order is already available. The new mean B_p enters the transverse source (5.28) undifferentiated.

Theorem 5.6. *There is an absolute exponent C such that the following holds if $\mathsf{D}^C\lambda^{-1/12} \le 1$. For all integers $n \ge 0$ and $p \ge 2$, the construction satisfies*

$$\|V_p\|_{[n],\gamma} + \|f_p + f_p^S\|_{[n],\gamma} + \lambda\|\mathsf{p}_p\|_{[n],\gamma} \le C_{n,p}\mathsf{D}^C\lambda^{-5/4}\lambda^{-2(p-1)/3}, \tag{5.34}$$

$$\|B_p\|_{[n]} + \|\bar{f}_p\|_{[n]} \le C_{n,p}\mathsf{D}^C\lambda^{-5/2}\lambda^{-2(p-2)/3}, \tag{5.35}$$

$$\|C_{p+1}\|_{[n],\gamma} \le C_{n,p}\mathsf{D}^C\lambda^{-9/4}\lambda^{-2(p-1)/3}. \tag{5.36}$$

The bounds (5.34) and (5.36) also hold for $p = 1$, with $f_1 = f_1^S = 0$. For all integers $P \ge 2$ and $m \ge 0$, the expansion of Proposition 5.4 at the depth P satisfies

$$\|W_{\mathrm{osc}}\|_{[m],\gamma} \le C_{m,P}\mathsf{D}^C\lambda^{-5/4}, \qquad \Big\|\sum_{p=2}^{P} B_p\Big\|_{[m]} + \Big\|\sum_{p=2}^{P} \bar{f}_p\Big\|_{[m]} \le C_{m,P}\mathsf{D}^C\lambda^{-5/2}, \tag{5.37}$$

and its remainder satisfies

$$\|R_P\|_{[m]} \le C_{m,P}\lambda^{-5/4}\lambda^{-2P/3}. \tag{5.38}$$

Proof. By (D.4), (D.3) and (5.8), since ℓ_*^{-1} is a power of D and $\kappa\partial_s$ costs $|\kappa|h^{-1} \le \mathsf{D}^2\lambda^{3/2}$ by (5.3), fields Z, Z', Y satisfy, with the weight on either factor of a product or on neither,

$$\begin{aligned} \|(Z\cdot\mathbf{d})Z'\|_{[n],\gamma} + \|(\mathbf{d}\cdot Z)Z'\|_{[n],\gamma} &\le C_n\mathsf{D}^C\|Z\|_{[n+1]}\|Z'\|_{[n+1],\gamma}, \\ \|(Z\cdot\kappa)\partial_s Y\|_{[n],\gamma} &\le C_n\mathsf{D}^C\lambda^{3/2}\|Z\|_{[n]}\|Y\|_{[n+1],\gamma}. \end{aligned} \tag{5.39}$$

For $p = 1$, (5.22) and (5.31) give (5.34) and (5.36) for A_1, $\mathsf{p}[A_1]$, $C[A_1]$ on $[t_0, T]$, and $0 \le \widetilde{\chi} \le 1$ gives them for V_1, p_1, C_2.

To keep powers of D from accumulating with the depth, we propagate a bound for the transverse source and the mean that contains only powers of λ. For $p \ge 2$ the induction claim at every order is

$$\|f_p + f_p^S\|_{[n],\gamma} + \lambda^{1/2}\|B_p\|_{[n]} \le C_{n,p}\lambda^{-5/4}\lambda^{-2(p-1)/3}. \tag{5.40}$$

For B_p this is (5.35) with $\lambda^{1/12}$ in place of D^C. Since (5.40) contains no power of D, every exponent of D below is a sum of a bounded number of the exponents of (5.12), (5.21), (5.22), (5.31), (5.32) and (5.39), and it depends neither on the level nor on the depth. The theorem holds with the largest of these exponents, since $\mathsf{D} \ge 2$. In the rest of the proof every bound holds up to a factor $C_{n,p}$ or $C_{m,P}$, which we do not write.

The bound (5.40) at a level $i \ge 2$ gives (5.34) and (5.36) for V_i, p_i and C_{i+1}. Indeed, by Proposition 5.4 the source $f_i + f_i^S$ vanishes for $t < t_0$ and outside $\operatorname{supp} g$, so (5.21) bounds V_i and $\lambda\mathsf{p}_i$ by D^C times the right side of (5.40), and (5.31) then bounds C_{i+1}. For the bounds at each level it remains to prove (5.40) and (5.35).

Let $p \ge 2$, and suppose that (5.40) holds at the levels $2 \le i < p$ and at every order. Then (5.34) and (5.36) hold for V_i and C_{i+1}, $1 \le i < p$. Recall $Y_i = V_i + C_i$ and $Z_i = Y_i + B_i$. The right side of (5.36) for C_i and the bound (5.40) for B_i are at most the right side of (5.34) for V_i. Since $\gamma \le 1$, it follows that, for $i < p$,

$$\|Y_i\|_{[n],\gamma} + \|Z_i\|_{[n]} \le \mathsf{D}^C\lambda^{-5/4}\lambda^{-2(i-1)/3}. \tag{5.41}$$

The source f_p in (5.27) uses only the fields V_i, p_i and B_i with $i < p$, and the corrections C_i with $i \le p$. Subtracting the phase mean of $(Z_i\cdot\mathbf{d})Z_j$ removes $(B_i\cdot\mathbf{d})B_j$, and every other product has a factor Y_i or Y_j, to which we apply the γ norm. By (5.32) and (5.34) for V_{p-1}, the linear term of H_p is at most $\mathsf{D}^C\lambda^{-9/4}\lambda^{-2(p-2)/3}$; for $p = 2$ this is the bound for A_1, with $f = 0$ and $\widetilde{\chi} \le 1$. By (5.39) and (5.41), a product $(Z_i\cdot\mathbf{d})Z_j$ with $i + j = p$ is at most $\mathsf{D}^C\lambda^{-5/2}\lambda^{-2(p-2)/3}$. By (5.39), (5.41) and (5.36) for C_i, a product $(C_i\cdot\kappa)\partial_s Y_j$ with $i + j = p + 1$ is at most $\mathsf{D}^C\lambda^{3/2}\lambda^{-9/4}\lambda^{-5/4}\lambda^{-2(p-2)/3} =$

$\mathsf{D}^C\lambda^{-2}\lambda^{-2(p-2)/3}$. The right side of (5.40) is $\lambda^{-23/12}\lambda^{-2(p-2)/3}$, so the three bounds are this number times

$$\mathsf{D}^C\lambda^{-1/3}, \qquad \mathsf{D}^C\lambda^{-7/12}, \qquad \mathsf{D}^C\lambda^{-1/12},$$

respectively. Thus $\|f_p\|_{[n],\gamma}$ is at most $\mathsf{D}^C\lambda^{-1/12}$ times the right side of (5.40).

In the mean source $\bar f_p$, integration by parts in s removes the fast derivative just as at level two. By (3.6), for $i \ge 2$ we have

$$\langle (C_i\cdot\kappa)\partial_s Y_j\rangle_s = \langle (\mathbf{d}\cdot V_{i-1})Y_j\rangle_s. \tag{5.42}$$

The linear terms have zero phase average because C_p and p_{p-1} do. Products of one mean and one oscillation also average to zero, since B_i and κ are phase independent and $\langle Y_j\rangle_s = 0$. Thus the mean source is

$$-\bar f_p = \sum_{i+j=p}\langle (Y_i\cdot\mathbf{d})Y_j\rangle_s + \sum_{i+j=p}\langle (\mathbf{d}\cdot V_i)Y_j\rangle_s + \sum_{i+j=p}(B_i\cdot\mathbf{d})B_j,$$

with the additional term $R_U\circ X$ at $p=2$. By (5.39) and (5.41), the first two sums are at most $\mathsf{D}^C\lambda^{-5/2}\lambda^{-2(p-2)/3}$. By (5.39) and (5.40) at the lower levels, the third is at most $\mathsf{D}^C\lambda^{-7/2}\lambda^{-2(p-2)/3}$, which is smaller. The term $R_U\circ X$ is at most $C_n\lambda^{-5/2}$, by (5.10). Hence $\|\bar f_p\|_{[n]} \le \mathsf{D}^C\lambda^{-5/2}\lambda^{-2(p-2)/3}$ at every order. With this estimate at the order $n+1$, (5.12) bounds $\|B_p\|_{[n]}$ by D^C times as much. This proves (5.35) at the level p, and with the hypothesis $\mathsf{D}^C \le \lambda^{1/12}$ it gives (5.40) for B_p.

For f_p^S in (5.28), the means B_i, $2\le i\le p$, are now bounded by (5.40), and Y_{p+1-i} is bounded by (5.41). By (5.39), each product $(B_i\cdot\kappa)\partial_s Y_{p+1-i}$ is therefore at most $\mathsf{D}^C\lambda^{-3/2}\lambda^{-2(p-1)/3}$, which is $\mathsf{D}^C\lambda^{-1/4}$ times the right side of (5.40). With the bound for f_p, $\|f_p + f_p^S\|_{[n],\gamma}$ is at most $\mathsf{D}^C\lambda^{-1/12}$ times that right side, and the hypothesis gives (5.40) at the level p. This completes the induction. The number of products in a level and the constants of the lower levels at higher orders enter $C_{n,p}$.

Since $\lambda \ge 1$, summing (5.34)–(5.36) over $p \le P$ gives (5.37). For the remainder, recall $Y_{P+1} = Z_{P+1} = C_{P+1}$. By (5.36) for C_{P+1}, the bound (5.41) holds for $i \le P+1$. The products in (5.30) are therefore bounded as those of H_p and f_p^S were: the products $(Z_i\cdot\mathbf{d})Z_j$ with $i+j=q$, and $(C_i\cdot\kappa)\partial_s Y_j$ and $(B_i\cdot\kappa)\partial_s Y_j$ with $i+j=q+1$, are at most $\mathsf{D}^C\lambda^{-1/12}$ times $\lambda^{-5/4}\lambda^{-2(q-1)/3}$, and in (5.30) they have $q \ge P+1$. By (5.32) and (5.34) for V_P, the linear term $(\partial_t + M)C_{P+1} + \mathbf{d}\mathsf{p}_P$ is at most $\mathsf{D}^C\lambda^{-1/3}$ times $\lambda^{-5/4}\lambda^{-2P/3}$. The products of two means are bounded in the unweighted norm, and every other term in the weighted norm, which dominates the unweighted one because $\gamma \le 1$. With the hypothesis, this is (5.38). □

5.6. **The approximate solution.** Fix an integer $P \ge 2$. The profiles W, P of Proposition 5.4 at depth P give the Eulerian velocity and pressure

$$V = U + W^\natural, \qquad \Pi = p_U + \mathsf{P}^\natural, \qquad W = W_{\mathrm{osc}} + \bar W, \quad \bar W = \sum_{p=2}^{P} B_p.$$

Fix an absolute integer $b \ge 1$, at least as large as the exponents C of Theorem 5.6 and of Lemma 5.7, and sufficiently large for the finitely many estimates of this subsection and the next. Put

$$C_{\mathrm{step}} = \Gamma_P\mathsf{D}^b,$$

where $\Gamma_P \ge 1$ depends only on P and the fixed profiles. We take it sufficiently large compared with every $C_{2,P}$ in this subsection and the next. The estimates of the residual, the pressure Hessian and the flow use the profile bounds through order two. The truncation uses the velocity bound in $C^{P,1/2}$ and the tail estimate (B.2). From here on we assume

$$C_{\mathrm{step}}\lambda^{-1/12} \le 1. \tag{5.43}$$

Since $C_{\mathrm{step}} \ge \mathsf{D}^C$ for the exponent C of Theorem 5.6, the hypothesis of that theorem holds.

To estimate the velocity and residual in Eulerian coordinates, we evaluate each profile at $a = X(t,\cdot)^{-1}(x)$ and $s = k_* \cdot a$.

Lemma 5.7. *Let $g_0(t,a,s)$ be smooth and independent of the phase at the labels outside* $\operatorname{supp} g$. *Assume its phase average extends smoothly and periodically in a across the cell faces. There is an absolute exponent C such that, for every integer $m \geq 0$ and every $t \in [t_0, T]$,*

$$\sum_{|\beta|=m} \|\partial_x^\beta g_0^\natural(t)\|_{L^\infty} \leq C_m (\mathsf{D}^C \lambda^{3/2})^m \gamma(t) \|g_0\|_{[m],\gamma}, \tag{5.44}$$

$$\|g_0^\natural(t)\|_{C^{m,1/2}} \leq C_m (\mathsf{D}^C \lambda^{3/2})^{m+1} \gamma(t) \|g_0\|_{[m+1],\gamma}. \tag{5.45}$$

Both bounds also hold on $[0,T]$ with γ replaced by one, and for a phase-independent field with $\mathsf{D}^C\lambda^{3/2}$ replaced by D^C.

Proof. The field $\widetilde{\mathbf{d}}_i g_0$ is again independent of the phase outside $\operatorname{supp} g$, which lies in the interior of the cell, so $g_0^\natural$ and its derivatives extend smoothly and periodically. The spatial chain rule in (3.1) and $\widetilde{\mathbf{d}} = (D_a X)^{-T}(\nabla_a + k_* \partial_s)$ give

$$\sum_i \|\widetilde{\mathbf{d}}_i g_0\|_{[n],\gamma} \leq C_n \mathsf{D}^C \lambda^{3/2} \|g_0\|_{[n+1],\gamma}, \qquad n \geq 0,$$

by Lemma D.1 and (5.8): a label derivative costs ℓ_*^{-1}, a fixed power of D, and $k_*\partial_s$ costs $|k_*|h^{-1} \leq \mathsf{D}\lambda^{3/2}$, by (5.3). Iterate the displayed bound; at each time the supremum of a field is at most $\gamma(t)$ times its weighted norm of order zero. This proves (5.44). The bound $\|v\|_{C^{0,1/2}} \leq C(\|v\|_\infty + \|\nabla v\|_\infty)$, applied to derivatives through order m, proves (5.45) with one additional derivative. A phase-independent field has $\widetilde{\mathbf{d}} g_0 = \mathbf{d} g_0$, proving the last assertion. □

In Section 7 the oscillations are summed in the supremum norm, while their higher derivatives control the Fourier tail in (7.12). The means can extend across the torus and back to time zero, so their sum must converge at every fixed derivative order.

Lemma 5.8. *The field V is smooth, odd and divergence free, Π is smooth, and*

$$V_t + V \cdot \nabla V + \nabla \Pi = R_V, \qquad R_V = R_P^\natural + R_{\widetilde{\chi}}^\natural. \tag{5.46}$$

The oscillation satisfies

$$\|W_{\mathrm{osc}}^\natural\|_{C_t L^\infty} \leq C_{0,P} \mathsf{D}^b \lambda^{-5/4}, \tag{5.47}$$

and for every $m \geq 0$,

$$\begin{gathered} \|W_{\mathrm{osc}}^\natural\|_{C_t C^{m,1/2}} \leq C_{m,P} \mathsf{D}^{b(m+2)} \lambda^{3(m+1)/2 - 5/4}, \qquad \|\bar{W}^\natural\|_{C_t C^{m,1/2}} \leq C_{m,P} \mathsf{D}^{b(m+2)} \lambda^{-5/2}, \\ \|R_V\|_{C_t C^{1,1/2}} \leq C_{\mathrm{step}} \lambda^2 \big(\lambda^{-2P/3} + \|\widetilde{\chi}' \gamma\|_\infty\big), \end{gathered} \tag{5.48}$$

where the last supremum is over $[t_0, T]$.

Proof. By (3.1), at $x = X(t,a)$ one has $(\partial_t + U \cdot \nabla) W^\natural = (W_t)^\natural$, $W^\natural \cdot \nabla U = (MW)^\natural$, $W^\natural \cdot \nabla W^\natural = ((W \cdot \widetilde{\mathbf{d}})W)^\natural$ and $\nabla \mathsf{P}^\natural = (\widetilde{\mathbf{d}}\mathsf{P})^\natural$. Adding the definition of R_U and (5.29) gives (5.46), because $(R_U \circ X)^\natural = R_U$: the residual of the parent is canceled exactly. Also $\widetilde{\mathbf{d}} \cdot W = 0$ gives $\operatorname{div} V = 0$. Since $\operatorname{supp} g$ lies in the interior of the cell, the oscillatory terms in $W^\natural$ and $\mathsf{P}^\natural$ extend smoothly and periodically for every k_*. Proposition 5.4 gives the smoothness of V and Π, and $W^\natural$ is odd by the symmetry of the construction of Subsection 5.5 under $(a,s) \mapsto (-a,-s)$.

Since $\gamma \leq 1$, the supremum of $W_{\mathrm{osc}}^\natural$ is at most $\|W_{\mathrm{osc}}\|_{[0],\gamma}$, and (5.37) gives (5.47). The bound (5.45) of Lemma 5.7, applied to W_{osc} and $\bar{W}$ with (5.37) at the order $m+1$, proves the first two bounds of (5.48). By Proposition 5.4, R_P is independent of the phase outside $\operatorname{supp} g$, so (5.45) at $m = 1$, with γ replaced by one, and (5.38) bound $R_P^\natural$ in $C_t C^{1,1/2}$ by $C_{2,P} \mathsf{D}^C \lambda^{7/4} \lambda^{-2P/3}$. By (5.22)

and (5.31), $\|A_1 + C[A_1]\|_{[2],\gamma} \le C\mathsf{D}^C\lambda^{-5/4}$. At each time t, $R_{\widetilde{\chi}}$ is $\widetilde{\chi}'(t)$ times this field, so (5.45) at that time bounds $R^\natural_{\widetilde{\chi}}(t)$ in $C^{1,1/2}$ by $C\mathsf{D}^C\lambda^{7/4}|\widetilde{\chi}'(t)|\gamma(t)$. This proves the last bound of (5.48). □

In (5.48) the depth P controls the expansion remainder. For a later wave, (6.8) bounds γ on the interval where the time cutoff varies.

The next mean equation requires an upper bound on the added pressure Hessian, since the pressure Hessian enters the displacement form (4.7) with a minus sign. The next wave also needs the velocity gradient at the origin. Once $\widetilde{\chi} = 1$, the rank-one term in (5.50) supplies the wave contribution to the gradient in (6.5).

Lemma 5.9. *On* $[0,T]\times\mathbb{T}^3$,

$$D_x^2\mathsf{P}^\natural \le C_{\mathrm{step}}\lambda^{-1/4}\,\mathrm{Id}\,. \tag{5.49}$$

At $x = 0$, *for* $t_0 \le t \le T$,

$$\nabla V(t,0) = \nabla U(t,0) + \widetilde{\chi}(t)\,\mathbf{a}(t,0)\otimes\kappa(t,0) + E_c(t), \qquad |E_c| \le \lambda^{-1/3}. \tag{5.50}$$

Proof. Write the leading pressure as $\mathsf{p}_1 = -c_{\mathbf{a}}P_h$, with $c_{\mathbf{a}} = 2\widetilde{\chi}g(\kappa\cdot M\mathbf{a})/|\kappa|^2$. Since $D_x^2\mathsf{p}_1^\natural = (\widetilde{\mathbf{d}}\,\widetilde{\mathbf{d}}\,\mathsf{p}_1)^\natural$ and $\mathbf{d}_i\kappa_j = \mathbf{d}_j\kappa_i$, with $a = X(t,\cdot)^{-1}(x)$ and $s = k_*\cdot a$ on the right,

$$D_x^2\mathsf{p}_1^\natural = -c_{\mathbf{a}}F_h'\kappa\otimes\kappa + (\mathbf{d}\partial_s\mathsf{p}_1)\otimes\kappa + \kappa\otimes(\mathbf{d}\partial_s\mathsf{p}_1) + (\partial_s\mathsf{p}_1)\mathbf{d}\kappa + \mathbf{d}^2\mathsf{p}_1. \tag{5.51}$$

The product $\widetilde{\chi}g$ takes values in $[0,1]$ and vanishes off $[t_0,T]\times\operatorname{supp} g$, where $\kappa\cdot M\mathbf{a} > 0$ by (5.19). Hence $0 \le c_{\mathbf{a}}|\kappa|^2 \le 2|\kappa||M||\mathbf{a}|$. With $F_h' \ge -Ch$, the first tensor on the right is therefore at most $Chc_{\mathbf{a}}|\kappa|^2\,\mathrm{Id}$. On $[t_0,T]\times\operatorname{supp} g$, (5.19) gives $|\mathbf{a}| \le 2\lambda^{-3/4}$, while $|\kappa| \le \mathsf{D}^2\lambda$ and $|M| \le \mathsf{D}$ by (5.3) and (5.1). So $c_{\mathbf{a}}|\kappa|^2 \le 4\mathsf{D}^3\lambda^{1/4}$ and, since $h = \lambda^{-1/2}$, the first tensor is at most $C\mathsf{D}^3\lambda^{-1/4}\,\mathrm{Id}$. By (D.4), (D.3) and (5.8), the sum of the remaining tensors is at most $C\mathsf{D}^C\lambda^{3/2}\|\mathsf{p}_1\|_{[2]}$ in supremum norm, and $\lambda\|\mathsf{p}_1\|_{[2]} \le C\mathsf{D}^C\lambda^{-5/4}$ by (5.22). So their sum is at most $C\mathsf{D}^C\lambda^{-3/4}$.

For the higher oscillatory pressures, (5.34) gives the least decay in λ at $p = 2$, where it bounds p_2 by $C_{2,P}\mathsf{D}^C\lambda^{-1-23/12}$. Two Eulerian derivatives in (5.44) would cost λ^3, leaving $\lambda^{1/12}$. To bound the first derivative, use the transverse equation (5.16). For $2 \le p \le P$ it gives

$$\nabla_x\mathsf{p}_p^\natural = (\mathbf{d}\mathsf{p}_p + \kappa\partial_s\mathsf{p}_p)^\natural, \qquad \kappa\partial_s\mathsf{p}_p = P_\kappa(f_p + f_p^S - 2MV_p).$$

The label derivative costs a power of D, and the phase term is bounded by the source and velocity in (5.34). The coefficient estimates therefore give

$$\Big\|\widetilde{\mathbf{d}}\sum_{p=2}^{P}\mathsf{p}_p\Big\|_{[1],\gamma} \le C_{2,P}\mathsf{D}^C\lambda^{-23/12}.$$

The oscillatory pressures are supported in $\operatorname{supp} g$ in Lagrangian coordinates. One further Eulerian derivative costs $\lambda^{3/2}$ up to a power of D, so (5.44) gives

$$\Big\|D_x^2\sum_{p=2}^{P}\mathsf{p}_p^\natural\Big\|_\infty \le C_{2,P}\mathsf{D}^C\lambda^{-5/12}. \tag{5.52}$$

For the mean pressures, add the mean equations (5.11) of the levels $2 \le p \le P$. In Eulerian coordinates the sum is the linearized Euler equation (4.12) for $\bar{W}^\natural$ over U, with the pressure $\sum_p \bar{q}_p^\natural$ and the source $(\sum_p \bar{f}_p)^\natural$, and its divergence is

$$\Delta\sum_{p=2}^{P}\bar{q}_p^\natural = \operatorname{div}\Big(\sum_{p=2}^{P}\bar{f}_p\Big)^\natural - 2\operatorname{tr}\big(\nabla U\,\nabla\bar{W}^\natural\big).$$

The operator $D^2\Delta^{-1}$ is bounded on $C^{0,1/2}$. By (5.45) for phase-independent fields and (5.37), the source and $\bar{W}^\natural$ are at most $C_{2,P}\mathsf{D}^C\lambda^{-5/2}$ in $C_tC^{1,1/2}$, and $\|\nabla U\|_{C_tC^{0,1/2}} \le C\mathsf{D}\,\|\nabla U\|_\infty \le C\mathsf{D}^2$ by

(B.1) and (5.1). Hence the Hessian of the sum of the mean pressures is at most $C_{2,P}\mathsf{D}^C\lambda^{-5/2}$ in supremum norm. Together with the bounds for the tensors of (5.51) and with (5.52), this proves (5.49).

To compute the velocity gradient at $x = 0$, evaluate at $a = 0$ and $s = 0$, since the flow of U fixes the origin. As $g(0) = F_h'(0) = 1$ and $F_h(0) = 0$, the chain rule (3.1) gives $D_x V_1^\natural(t,0) = \widetilde{\chi}(t)\mathbf{a}(t,0)\otimes\kappa(t,0)$. Hence

$$E_c(t) = D_x(W_{\mathrm{osc}} - V_1)^\natural(t,0) + D_x\bar{W}^\natural(t,0).$$

The field $W_{\mathrm{osc}} - V_1 = \sum_{p=2}^{P} V_p + \sum_{p=2}^{P+1} C_p$ is supported in $\operatorname{supp} g$. By (5.34) and (5.36), its norm $\|\cdot\|_{[1],\gamma}$ is at most $C_{2,P}\mathsf{D}^C\lambda^{-23/12}$, the bound for V_2, and $\|\bar{W}\|_{[1]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/2}$ by (5.37). The bound (5.44) at $m = 1$ therefore gives $|E_c| \le C_{2,P}\mathsf{D}^C\lambda^{-5/12} = C_{2,P}\mathsf{D}^C\lambda^{-1/12}\cdot\lambda^{-1/3}$. Since b is at least the exponent C of this bound and $\Gamma_P \ge C_{2,P}$, (5.43) makes it at most $\lambda^{-1/3}$. This proves (5.50). □

To allow for the difference introduced by frequency truncation, let Y be the flow of $V + v$ for a small divergence-free field v. A direct Gronwall estimate for D_aY puts the large velocity gradient in an exponential. Write $Y(t,a) = X(t,\mathcal{A}_t(a))$ and let the label and phase evolve as independent variables. Incompressibility expresses the oscillatory phase velocity through a label divergence, giving a small velocity gradient in the scaled variables. We then differentiate the change of label on the graph $s = k_*\cdot a$ to recover $D\mathcal{A}_t$.

Lemma 5.10. *Let v be a smooth divergence-free field on $[0,T]\times\mathbb{T}^3$ with $\|v\|_{C_tC^2} \le \lambda^{-5/2}$. The flow Y of $V + v$ from time zero satisfies*

$$\sup|(D_aY)^{\pm1}| \le C_{\mathrm{step}}\lambda^{1/4}. \tag{5.53}$$

Proof. The field $v\circ X$ is independent of the phase, and $\mathbf{d}\cdot(v\circ X) = (\operatorname{div} v)\circ X = 0$, so $\widetilde{\mathbf{d}}\cdot(W + v\circ X) = 0$. By the chain rule and (5.8) for D_aX, $\|v\circ X\|_{[2]} \le C\mathsf{D}^C\|v\|_{C_tC^2}$. Put $c = (D_aX)^{-1}(W + v\circ X)$ and write $c = c_{\mathrm{osc}} + \bar{c}$ for its oscillatory and phase-independent parts, with c_{osc} repeated over the lattice by $(a,s)\mapsto(a+n, s+k_*\cdot n)$, $n\in\mathbb{Z}^3$. On the graph $s = k_*\cdot a$, the field c is then $(D_aX)^{-1}((W^\natural + v)\circ X)$. The Piola identity turns the solenoidality of $W + v\circ X$ into $(\nabla_a + k_*\partial_s)\cdot c = 0$, and the zero-mean phase primitive of its oscillatory part gives $k_*\cdot c_{\mathrm{osc}} = -\partial_s^{-1}\operatorname{div}_a c_{\mathrm{osc}}$. With (5.37) and (5.8), the norms without weight therefore satisfy

$$\|c_{\mathrm{osc}}\|_{[2]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/4}, \qquad \|\bar{c}\|_{[2]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/2}, \qquad \|k_*\cdot c\|_{[1]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/4};$$

in the last bound the oscillatory part costs one label derivative and the mean part the factor $|k_*| \le \mathsf{D}\lambda$ of (5.3).

Let $\mathscr{L}_t$, with first component $a + \eta_a(t,a,s)$, be the flow from time zero of

$$\dot{a} = c(t,a,s), \qquad \dot{s} = k_*\cdot c(t,a,s). \tag{5.54}$$

In the coordinates $(a/\ell_*, s/h)$ this vector field is $(c/\ell_*, k_*\cdot c/h)$, and its gradient is at most $C(\ell_*^{-1}\|c\|_{[1]} + h^{-1}\|k_*\cdot c\|_{[1]}) \le C_{2,P}\mathsf{D}^C\lambda^{-3/4} \le C_{2,P}\mathsf{D}^C\lambda^{-1/12}$. Since b is at least the exponent C of this bound, (5.43) makes it at most $C_{2,P}/\Gamma_P$, a small absolute constant since Γ_P is large. Since $t \le T \le 1$, Gronwall's inequality then bounds the gradients of $\mathscr{L}_t$ and of $\mathscr{L}_t^{-1}$ in these coordinates by 2. The scaled gradient of c is at most $\|c\|_{[1]}$, so integrating $\dot{\eta}_a = c$ along the flow gives

$$\ell_*|\partial_a\eta_a| + h|\partial_s\eta_a| \le C\|c\|_{[1]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/4},$$

and the displacement of $\mathscr{L}_t^{-1}$ satisfies the same bound.

Differentiating $Y = X(t,\mathcal{A})$ gives $\partial_t\mathcal{A} = ((D_aX)^{-1}((W^\natural + v)\circ X))(t,\mathcal{A})$. Since $s - k_*\cdot a$ is preserved by (5.54), the graph $s = k_*\cdot a$ is invariant, so $\mathcal{A}_t(a) = a + \eta_a(t,a,k_*\cdot a)$, the first component

of $\mathscr{L}_t$ on this graph. The same formula with the displacement of $\mathscr{L}_t^{-1}$ gives the inverse map. Thus $D\mathcal{A}_t - \mathrm{Id} = \partial_a\eta_a + \partial_s\eta_a \otimes k_*$, and with the corresponding identity for the inverse,

$$|D\mathcal{A}_t^{\pm 1} - \mathrm{Id}\,| \le \big(\ell_*^{-1} + |k_*|h^{-1}\big)C_{2,P}\mathsf{D}^C\lambda^{-5/4} \le C_{2,P}\mathsf{D}^C\lambda^{1/4}.$$

Then $D_aY = (D_aX \circ \mathcal{A})D\mathcal{A}$ and $(D_aY)^{-1} = (D\mathcal{A})^{-1}((D_aX)^{-1} \circ \mathcal{A})$, with the bound (5.1) for the deformation of the parent, give (5.53). □

5.7. **Frequency truncation.** We truncate V with the smooth frequency cutoff S_Λ of Appendix B, as in the mollification step of [6, Section 2.4]. The cutoff bounds all higher derivatives of the next parent in terms of Λ and its supremum. We choose its pressure p so that its residual is divergence free. The equation for the expansion pressure Π contains $\operatorname{div} R_V$ by (5.46), so the bound for $D^2(p-\Pi)$ includes both the residual of V and the Fourier tail. The iteration uses the gradient bound on the new residual and the upper bound on $D^2(p-p_U)$ to propagate (5.2).

Proposition 5.11. *Let $\Lambda \ge 1$, and with the depth P of the expansion put*

$$u = S_\Lambda V, \qquad v_\Lambda = u - V, \qquad \varepsilon_\Lambda = \Lambda^{2-P}\big(1 + \|V\|_{C_tC^{P,1/2}}\big)^2. \tag{5.55}$$

Let p be the function of zero mean with $\Delta p = -\operatorname{tr}((\nabla u)^2)$, and let $R_u = u_t + u\cdot\nabla u + \nabla p$. Then u is smooth, odd, divergence free and supported on frequencies at most 2Λ, its residual R_u is odd, divergence free and supported on frequencies at most 4Λ, and

$$\|v_\Lambda\|_{C_tC^{2,1/2}} \le C_{2,P}\varepsilon_\Lambda, \qquad \|R_u\|_{C_tC^{1,1/2}} + \big\|D^2(p-\Pi)\big\|_{C_tC^{0,1/2}} \le C\,\|R_V\|_{C_tC^{1,1/2}} + C_{2,P}\varepsilon_\Lambda. \tag{5.56}$$

Moreover, on $[0,T]\times\mathbb{T}^3$,

$$D^2(p-p_U) \le \big(C_{\mathrm{step}}\lambda^{-1/4} + C\,\|R_V\|_{C_tC^{1,1/2}} + C_{2,P}\varepsilon_\Lambda\big)\,\mathrm{Id}\,. \tag{5.57}$$

Proof. By Lemma B.1, u and v_Λ are odd and divergence free and u is supported on frequencies at most 2Λ. Since $\operatorname{div}(u\cdot\nabla u) = \operatorname{tr}((\nabla u)^2)$, the residual is $R_u = \partial_t u + \mathbb{P}\operatorname{div}(u\otimes u)$; it is odd, divergence free and supported on frequencies at most 4Λ. Applying S_Λ and then $\mathbb{P}$ to (5.46) gives $\partial_t u + \mathbb{P}\operatorname{div} S_\Lambda(V\otimes V) = \mathbb{P}S_\Lambda R_V$. Subtracting this from the formula for R_u gives

$$R_u = \mathbb{P}\big[S_\Lambda R_V + \operatorname{div}\big(v_\Lambda\otimes u + V\otimes v_\Lambda + (\mathrm{Id} - S_\Lambda)(V\otimes V)\big)\big]. \tag{5.58}$$

To bound R_u in $C^{1,1/2}$ we need the three quadratic tensors in (5.58) in $C^{2,1/2}$. This accounts for the power Λ^{2-P} and the square in ε_Λ. The tail bound (B.2) at the order P gives $\|v_\Lambda\|_{C^{s,1/2}} \le C_{2,P}\Lambda^{s-P}\|V\|_{C^{P,1/2}}$ for $s \le 2$, and in particular the first bound of (5.56). By Leibniz' rule and the algebra inequality for the norm of $C^{0,1/2}$, for every integer $m \ge 0$,

$$\|fg\|_{C^{m,1/2}} \le C_m\,\|f\|_{C^{m,1/2}}\,\|g\|_{C^{m,1/2}}\,. \tag{5.59}$$

Hence the three tensors on the right of (5.58) are at most $C_{2,P}\Lambda^{2-P}\|V\|^2_{C^{P,1/2}} \le C_{2,P}\varepsilon_\Lambda$ in $C^{2,1/2}$. The periodic multipliers $\mathbb{P}$ and $D^2\Delta^{-1}$ of order zero are bounded on $C^{0,1/2}$ and commute with derivatives, as does S_Λ; this bounds R_u in $C^{1,1/2}$. For the pressure, the divergence of (5.46) is $\Delta\Pi = \operatorname{div} R_V - \operatorname{tr}((\nabla V)^2)$. Hence

$$\Delta(p-\Pi) = -\operatorname{tr}(\nabla v_\Lambda\nabla V) - \operatorname{tr}(\nabla u\nabla v_\Lambda) - \operatorname{div} R_V,$$

and $\big\|D^2(p-\Pi)\big\|_{C^{0,1/2}} \le C\,\|v_\Lambda\|_{C^{1,1/2}}\,(\|V\|_{C^{1,1/2}} + \|u\|_{C^{1,1/2}}) + C\,\|R_V\|_{C^{1,1/2}}$. This proves (5.56). Since $p - p_U = (p-\Pi) + \mathsf{P}^\natural$, Lemma 5.9 then gives (5.57). □

6. The initial wave and the inductive step

The bounds (5.5) in Section 5 require a seed whose growth controls every transverse amplitude at the origin and whose first coupling stays positive. The first wave supplies the shear for the second at time zero. Each later wave grows in the shear of its predecessor. Its gain allows us to switch it on while its amplitude is still small compared with its maximum.

6.1. **The initial wave over a stationary shear.** For $0 < \rho \leq 1$ let

$$U_0(x) = \sin(2\pi x_2)(e_1 + \rho e_3), \qquad B_\rho = 2\pi(e_1 + \rho e_3) \otimes e_2 \tag{6.1}$$

be the stationary shear of Section 1 and its gradient at the origin. The field U_0 is smooth, odd, divergence free and supported on the Fourier modes $\pm e_2$. As it depends only on x_2 and has no second component, $U_0 \cdot \nabla U_0 = 0$. Its pressure and residual therefore vanish. Each particle keeps its second coordinate, so the flow and its deformation are

$$X_0(t,a) = a + t\sin(2\pi a_2)(e_1 + \rho e_3), \qquad (D_a X_0)^{\pm 1}(t,a) = \mathrm{Id} \pm t\cos(2\pi a_2) B_\rho. \tag{6.2}$$

The inverse formula uses $B_\rho^2 = 0$. The origin is a fixed particle and $|B_\rho| \leq 4\pi$.

Take the first wave of frequency λ over U_0 on $[0,T]$ with

$$k_* = \lambda e_1, \qquad \widetilde{\chi} \equiv 1, \tag{6.3}$$

so it is fully present at time zero. At the origin solve the seed equation (3.2) with initial direction e_2 and scale the solution so that $\max_{[0,T]} |\mathbf{a}(t,0)| = \lambda^{-3/4}$. At time zero $\kappa(0,a) = \lambda e_1$ at every label, so the projection in (5.4) leaves this initial amplitude unchanged. The initial frame at the origin is (e_1, e_2, e_3), in which

$$e_1 \cdot B_\rho e_2 = 2\pi, \qquad e_3 \cdot B_\rho e_2 = 2\pi\rho. \tag{6.4}$$

The second wave starts at time zero, so its amplification parameter is the ratio $\eta(0)/\nu(0) = \rho$ of these couplings in the first wave's frame. The shear that amplifies it is the leading gradient $\mathbf{a}(0,0) \otimes \kappa(0,0)$ contributed by the first wave in (5.50).

Lemma 6.1. *If $T|B_\rho| \leq 1/8$, the first wave satisfies the two conditions (5.5) for every $\mathsf{D} \geq 2$, and its strength at the origin at time zero lies between $\lambda^{1/4}/2$ and $\lambda^{1/4}$.*

Proof. At the origin the seed equation reads $\mathbf{b}_t = -(\mathrm{Id} - 2P_\kappa)B_\rho \mathbf{b}$. Since $\mathrm{Id} - 2P_\kappa$ is a reflection, every nonzero solution satisfies $\left|\partial_t \log|\mathbf{b}|\right| \leq |B_\rho|$. Applying this estimate to $\mathbf{b}$ and to the seed $\mathbf{a}$, whose length is proportional to γ, gives

$$\frac{|\mathbf{b}(t)|}{\gamma(t)} \leq e^{2T|B_\rho|}\frac{|\mathbf{b}(t')|}{\gamma(t')} \leq 2\frac{|\mathbf{b}(t')|}{\gamma(t')} \qquad (0 \leq t' \leq t \leq T).$$

The same estimate on the seed gives $e^{-T|B_\rho|}\lambda^{-3/4} \leq |\mathbf{a}(0,0)| \leq \lambda^{-3/4}$. Multiplying by $|\kappa(0,0)| = \lambda$ gives the initial strength.

For the sign of the coupling, the frame equations (2.2) give $|\partial_t E_1|, |\partial_t E_2| \leq |B_\rho|$. Thus the first coupling can change by at most $2t|B_\rho|^2$ from its value in (6.4), and

$$\nu(t,0) = E_1(t) \cdot B_\rho E_2(t) \geq 2\pi - 2t|B_\rho|^2 \geq \pi.$$

□

6.2. **Choosing the next wave.** Let U and its flow X be as in Subsection 5.1. The new wave starts at t_0 with frequency λ. The origin is a fixed particle. Once the preceding wave's cutoff equals one, (5.50) and the truncation estimate (5.56) give its contribution to the gradient there. Suppose, accordingly, that on $[t_0, T]$ the gradient $M = \nabla U(t,0)$ has the form

$$M = B + \mathbf{a}_p \otimes \kappa_p + E_c, \tag{6.5}$$

where B is the smooth trace-free gradient of the older velocity, $(\mathbf{a}_p, \kappa_p)$ is the preceding wave over B, and E_c contains the errors from its corrections and truncation. The preceding pair solves (2.1) with $D = B$. In its moving frame (E_i) write $B_{ij} = E_i \cdot B E_j$. At t_0 freeze its strength $\sigma = |\mathbf{a}_p||\kappa_p|$ and its couplings $\nu = B_{12} > 0$ and $\eta = B_{32} > 0$. As in Subsection 2.2.1, put $r = \sqrt{2\sigma\nu}$, $S = \sigma/r$, $\varepsilon = \eta/\nu$ and $\tau = r(t - t_0)$.

Choose the wave vector at activation perpendicular to the preceding wave vector and amplitude. Since $\kappa = (D_a X)^{-T} k_*$, the time-zero vector that gives $\kappa(t_0, 0) = \lambda E_3(t_0)$ is

$$k_* = \lambda (D_a X(t_0, 0))^T E_3(t_0).$$

For the amplitude take the growing direction $-E_1(t_0)/S + E_2(t_0)$. In the coordinates of (2.3) this is $(\alpha_1, \alpha_2) = (-1, 1)$ at $\tau = 0$. Scale the seed by a positive constant so that $\max_{[t_0,T]} |\mathbf{a}(t, 0)| = \lambda^{-3/4}$, and write $\gamma(t) = \lambda^{3/4} |\mathbf{a}(t, 0)|$ as in Subsection 5.1.

To apply the comparison of Subsection 2.2, we need to bound the variation of the two couplings of B in the moving frame. When $\varepsilon \le 1$, the comparison error (2.6) satisfies

$$\delta \le C \left[\frac{(T - t_0)\big(\|B\|_\infty^2 + \|B_t\|_\infty\big)}{\eta} + \frac{\|B\|_\infty}{r} + \frac{\|E_c\|_\infty}{\nu} \right]. \tag{6.6}$$

Here the suprema are over $[t_0, T]$. Indeed, the frame equations (2.2) give $|\partial_t E_i| \le 2 \|B\|_\infty$, so $|\partial_t B_{ij}| \le 4 \|B\|_\infty^2 + \|B_t\|_\infty$. Integration from t_0 bounds the changes in B_{12} and B_{32}. Dividing by their frozen values ν and η, with $\eta \le \nu$, gives the first term of (6.6). The other two terms occur directly in (2.6).

Suppose the normalized interval has length $r(T - t_0) \ge 2$ and that (2.7) holds with this length in place of T. Lemmas 2.6 and 2.7 then apply to the chosen direction. The positive rescaling of the seed preserves its frame and all ratios of its lengths. Write ν_+, η_+ for the couplings of M in the frame of the new wave. For $t_0 \le t' \le t \le T$ and every transverse solution $\mathbf{b}$ of the seed equation at the origin, the two lemmas give

$$\frac{|\mathbf{b}(t)|}{\gamma(t)} \le CS \, \frac{|\mathbf{b}(t')|}{\gamma(t')}, \qquad \nu_+(t) \ge \tfrac{1}{2} \nu \varepsilon. \tag{6.7}$$

These are the two estimates (5.5) when D bounds CS and $2/(\nu\varepsilon)$.

We switch on the wave during the first unit of normalized time. With the function χ of Subsection 5.1, take

$$\widetilde{\chi}(t) = 1 - \chi\big(1 + r(t - t_0)\big) \quad (t \ge t_0), \qquad \widetilde{\chi}(t) = 0 \quad (t < t_0).$$

This cutoff is smooth, equals one for $\tau \ge 1$ and satisfies $|\widetilde{\chi}'| \le Cr$. Prescribe the coupling ratio $0 < \varepsilon_+ \le \varepsilon/32$ for the following wave, and suppose $r(T - t_0) \ge 2(\varepsilon\varepsilon_+)^{-1/2}$. By (2.24),

$$\gamma \le \exp\big(3 - \varepsilon^{-1/2}\big) \qquad \text{on } [t_0, t_0 + 1/r]. \tag{6.8}$$

Thus the factor $\|\widetilde{\chi}' \gamma\|_\infty$ in the residual estimate (5.48) is at most $Cr \exp(3 - \varepsilon^{-1/2})$.

Proposition 2.8 also gives a time in $\tau_m \le \tau \le 2(\varepsilon\varepsilon_+)^{-1/2}$ at which $\eta_+/\nu_+ = \varepsilon_+$. We activate the following wave at the first such crossing in this interval. Since the reference amplitude peaks at $\tau_m \ge 2/\varepsilon > 1$, the cutoff already equals one there, and (2.23) gives the strength of the shear available to amplify that wave.

7. A common sequence of approximate solutions

We construct a single sequence of approximate solutions for all derivative orders. The parent for wave 1 is the stationary shear (6.1) with $\rho = \varepsilon_1$. At step j we choose the wave over U_{j-1} as in Section 6 and apply Proposition 5.11 to obtain U_j. The frequencies and orders are fixed by (1.14). The residual of U_j must already be small on the scale of the frequency λ_{j+1} of the next wave.

Write X_j for the flow of U_j from time zero, and π_j and R_j for its pressure and its residual, defined as p_U and R_U are for the parent in Subsection 5.1. At the center, let $\mathbf{a}_j, \kappa_j, \sigma_j$ be the seed, wave vector and strength of wave j, and let

$$\nu_j = \frac{\kappa_j}{|\kappa_j|} \cdot \nabla U_{j-1}(t, 0) \frac{\mathbf{a}_j}{|\mathbf{a}_j|}, \qquad \eta_j = \frac{\kappa_j \times \mathbf{a}_j}{|\kappa_j||\mathbf{a}_j|} \cdot \nabla U_{j-1}(t, 0) \frac{\mathbf{a}_j}{|\mathbf{a}_j|} \tag{7.1}$$

be the couplings of $\nabla U_{j-1}(t,0)$ in the frame of this pair. For $j \geq 2$ these are ν_+, η_+ of Subsection 6.2. Let $\widetilde{\chi}_j$ be the time cutoff of wave j and $O_j = W^{\natural}_{\mathrm{osc}}$ its oscillatory velocity increment before truncation, as in Lemma 5.8.

The first two waves start at time zero. Given the activation time t_j, we fix the final time S_j of the endpoint equations by

$$t_1 = t_2 = 0, \qquad \tau_j = \lambda_{j-1}^{-1/8} L_{j+1}^3, \qquad S_0 = 1, \qquad S_j = t_j + 2\tau_j \quad (j \geq 1).$$

For $j \geq 2$ we normalize the seed on $[t_j, S_j]$. The central wave equations over U_{j-1} then determine t_{j+1} on that interval. We take the first crossing of $\eta_j/\nu_j = \varepsilon_j$ in the interval of Proposition 2.8. It will occur by $t_j + \tau_j$, leaving room before S_j for the subsequent activation times. To make the remainder of the expansion small at frequency λ_{j+1}, we take the depth

$$P = 6K_{j+1}. \tag{7.2}$$

Theorem 7.1. *For every sufficiently large N there are fields U_j, $j \geq 1$, and times $0 = t_1 = t_2 < t_3 < \cdots$ with the following properties, in which U_0 is the shear (6.1) at $\rho = \varepsilon_1$:* (T1)–(T3) *for $j \geq 0$ and* (T4) *for $j \geq 1$.*

(T1) U_j is a smooth odd divergence-free field on $[0, S_j] \times \mathbb{T}^3$, supported on frequencies at most $2\Lambda_j$.

(T2) $\|U_j\|_{C_t L^\infty} \leq \lambda_j$, $\sup |(D_a X_j)^{\pm 1}| \leq \lambda_j$ and $\|R_j\|_{C_t L^\infty} \leq \lambda_{j+1}^{-3}$.

(T3) $D^2 \pi_j \leq \left(\sum_{i=1}^{j} \lambda_{i-1}^{-1}\right) \mathrm{Id}$ on $[0, S_j] \times \mathbb{T}^3$.

(T4) At the center on $[t_j, S_j]$, $\nabla U_j = \nabla U_{j-1} + \widetilde{\chi}_j \, \mathbf{a}_j \otimes \kappa_j + E_{c,j}$ with $|E_{c,j}| \leq \lambda_j^{-1/4}$. Moreover $0 \leq t_{j+1} - t_j \leq \tau_j$ and $\widetilde{\chi}_j = 1$ on $[t_{j+1}, S_j]$, so that the gradient of U_j has the splitting (6.5) there. At t_{j+1}, with an absolute constant $C \geq 2$ for which (2.23) holds,

$$\frac{\eta_j}{\nu_j} = \varepsilon_j, \qquad \pi \leq \nu_j \leq 4\pi, \qquad C^{-1} \lambda_j^{1/4} \leq \sigma_j \leq C \lambda_j^{1/4} L_{j+1}^2. \tag{7.3}$$

The activation times converge to a limit $T_\infty \in (0, \infty)$, and $T_\infty < S_j$ for every j. For $j \geq 1$ the oscillation O_j is supported, at each time of $[0, S_j]$, in the ball of radius λ_{j-1}^{-1} about the origin, and

$$\|O_j\|_{C([0,S_j];L^\infty)} \leq \lambda_j^{-1}, \qquad \|U_j - U_{j-1} - O_j\|_{C([0,S_j];C^{m,1/2})} \leq \lambda_j^{-1} \quad (0 \leq m \leq q_j). \tag{7.4}$$

For $j \geq 2$ the oscillation O_j vanishes on $[0, t_j]$.

For each fixed m, the orders q_j eventually exceed m. By (7.4), the increments $U_j - U_{j-1} - O_j$ are then summable in $C^{m,1/2}$ on the common interval $[0, T_\infty]$. On $[0, t_j]$ they give the entire velocity increment for $j \geq 2$. This is the estimate used for convergence in Section 8. Figure 4 shows the intervals on which the oscillations are present.

7.1. **The data of a step.** At any index j for which (T1) and (T2) hold, the frequency bound gives estimates for the gradient and its time derivative. As in Subsection 5.1, the residual is supported on frequencies at most $4\Lambda_j$ and

$$\partial_t U_j = -\mathbb{P} \operatorname{div}(U_j \otimes U_j) + R_j. \tag{7.5}$$

By (B.1), $\|\nabla U_j\|_\infty \leq C\Lambda_j \lambda_j$ and $\|\nabla R_j\|_\infty \leq C\Lambda_j \lambda_{j+1}^{-3}$. For the time derivative of the gradient, apply (B.1) to $U_j \otimes U_j$ and use the boundedness of $\mathbb{P}$ on $C^{0,1/2}$ in (7.5). Two spatial derivatives of the product in $C^{0,1/2}$ cost at most $C\Lambda_j^3 \lambda_j^2 = C\lambda_j^{17}$. Thus for large N,

$$\|\nabla U_j\|_\infty \leq \lambda_j^9, \qquad \|\partial_t \nabla U_j\|_\infty \leq \lambda_j^{18} \quad \text{on } [0, S_j]. \tag{7.6}$$

At step j we apply Section 5 with $U = U_{j-1}$, $\lambda = \lambda_j$, activation time t_j and final time S_j. Write $\mu = \lambda_{j-1}$ and $\widetilde{\chi} = \widetilde{\chi}_j$. By (7.6) at $j-1$ we can take $\mathsf{D} = \mu^9$ for the parent gradient. When $j \geq 2$ we also use (7.6) at $j-2$ for the older gradient in the wave comparison. The final Fourier cutoff is $\Lambda_j = \lambda^5$.

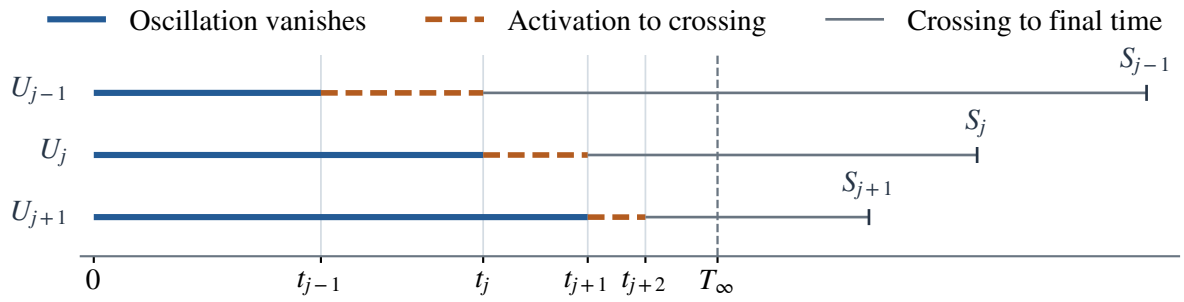


FIGURE 4. Three successive approximate solutions for $j \geq 4$. In each row the oscillation vanishes on the thick solid segment $[0, t_i]$. The dashed segment ends at the crossing t_{i+1}, when the next wave is activated. Each solution continues to its final time $S_i > T_\infty$.

To compare successive final times, observe that by (1.15), $\tau_j = \mu^{-1/8} L_{j+1}^3 \leq \mu^{-1/16}$ for large N. Applying this bound at $j+1$ and using $\lambda_j \geq \mu^4$ gives $4\tau_{j+1} \leq 4\mu^{-1/4} \leq \mu^{-1/8} \leq \tau_j$. Thus

$$\tau_j \leq \lambda_{j-1}^{-1/16}, \qquad 4\tau_{j+1} \leq \tau_j, \tag{7.7}$$

uniformly in j for large N.

Finally the depth $P = 6K_{j+1}$ in (7.2) requires estimates through order $P+1$. Since $P+1 \leq 7K(q_{j+1})$, $\mathcal{B}(q_{j+1})$ bounds all the fixed-order constants used in Section 5 and Lemma B.1 at these orders and this depth. The choice of q_{j+1} in (1.14) gives $\mathcal{B}(q_{j+1}) \leq \mu$. With $\mathsf{D} = \mu^9$ and $\lambda = \mu^{K_j}$, $K_j = 320b(q_j+1)$, we obtain

$$\mathcal{B}(q_{j+1})\mathsf{D}^b \leq \mu^{10b} \leq \lambda^{1/32}. \tag{7.8}$$

7.2. **Proof of Theorem 7.1.** For $j = 0$, the shear U_0 has only the Fourier modes $\pm e_2$, $\|U_0\|_\infty \leq 2$, and its pressure and residual vanish. Its deformation and inverse have norm at most $1 + 4\pi$ on $[0, S_0]$ by (6.2). These give (T1)–(T3) for large N.

Suppose the construction has been carried out through step $j-1$. For $j = 1$, (7.7) gives $S_1 = 2\tau_1 \leq S_0 = 1$. For $j \geq 2$, the previous crossing occurred by $t_{j-1} + \tau_{j-1}$. Since $2\tau_j \leq \tau_{j-1}/2$,

$$S_j = t_j + 2\tau_j \leq t_{j-1} + \tfrac{3}{2}\tau_{j-1} < S_{j-1}.$$

Thus U_{j-1} is available on the entire interval $[0, S_j]$ of the endpoint problem.

(1) *Estimates for the parent.* On $[0, S_j] \subset [0, S_{j-1}]$, (T1) bounds the frequencies of $U = U_{j-1}$ by $2\Lambda_{j-1} = 2\mu^5 \leq 2\mathsf{D}$. By (T2) and (7.6), $\sup |(D_a X)^{\pm 1}| \leq \mu$ and $\|\nabla U\|_\infty \leq \mu^9 = \mathsf{D}$, as required in (5.1). The residual bound $\|R_{j-1}\|_{C_t L^\infty} \leq \lambda^{-3}$ in (T2) gives (5.6). For the endpoint problem we also need (5.2). The derivative bound preceding (7.6) gives $\|\nabla R_{j-1}\|_\infty \leq C\mu^5\lambda^{-3}$. In (T3) the sum is at most $2/N$, since $\lambda_0 = N$ and $\lambda_i \geq 2\lambda_{i-1}$. Thus $D^2\pi_{j-1} - \nabla R_{j-1} \leq (2/N + C\mu^5\lambda^{-3})\,\mathrm{Id}$ as quadratic forms, which gives (5.2) for large N.

(2) *Choosing wave j.* For $j = 1$ take the wave (6.3) of frequency λ_1 on $[0, S_1]$. Since $4\pi S_1 \leq 1/8$ for large N, Lemma 6.1 gives (5.5) and the strength bounds in (7.3) at $t_2 = t_1 = 0$. The couplings there are $\nu_1 = 2\pi$ and $\eta_1 = 2\pi\varepsilon_1$ by (6.4). The first wave has $\widetilde{\chi}_1 \equiv 1$.

For $j \geq 2$, (T4) at $j-1$ gives the splitting (6.5) on $[t_j, S_j]$ with $B = \nabla U_{j-2}(t, 0)$, the preceding pair $(\mathbf{a}_{j-1}, \kappa_{j-1})$ and $\|E_c\|_\infty \leq \mu^{-1/4}$. The pair solves (2.1) over B because the flow of U_{j-2} fixes the origin. Freeze its couplings and strength at t_j. For $j = 2$ these are the values of wave 1 at time zero. For $j \geq 3$ they are the values at the preceding crossing. In both cases (7.3) gives

$$\varepsilon = \varepsilon_{j-1} = L_j^{-2}, \qquad \pi \leq \nu \leq 4\pi, \qquad c\mu^{1/4} \leq \sigma \leq C\mu^{1/4} L_j^2.$$

Choose the wave of Subsection 6.2, whose wave vector has length λ at t_j. Since $r = \sqrt{2\sigma\nu}$, $c\mu^{1/8} \le r \le C\mu^{1/8}L_j$. The normalized length $r(S_j - t_j) = 2r\mu^{-1/8}L_{j+1}^3$ is therefore between 2 and CL_{j+1}^4 for large N.

To check the comparison on this interval, (7.6) at $j-2$ gives $\|B\|_\infty \le \lambda_{j-2}^9 \le \mu^{1/32}$ and $\|B_t\|_\infty \le \mu^{1/16}$, since $\lambda_{j-2} = \mu^{1/K_{j-1}}$ and $K_{j-1} > 288$. With $\eta = \nu\varepsilon$, the three terms in (6.6) give

$$\delta \le C\big(\varepsilon^{-1}L_{j+1}^3\mu^{-1/16} + \mu^{-3/32} + \mu^{-1/4}\big).$$

By (1.15), $L_j \le L_{j+1} \le (\log\mu)^2$. Thus $C_0(2r\tau_j)^{C_0}\delta/\varepsilon$ is bounded by $\mu^{-1/16}$ times a fixed power of $\log\mu$, and $C_0\varepsilon \le C_0(\log N)^{-2}$. For large N both conditions of (2.7) hold. In the resulting central estimates (6.7), $CS = Cr/(2\nu) \le C\mu^{1/8}L_j$ and $2/(\nu\varepsilon) \le 2L_j^2/\pi$. Both are at most $\mu \le \mathsf{D}$ for large N. This proves (5.5) for wave j.

(3) *The next activation time.* For $j \ge 2$ we seek a crossing with $\eta_j/\nu_j = \varepsilon_j$ in the first half of $[t_j, S_j]$. Then $t_{j+1} \le t_j + \tau_j$ leaves room for the next interval, since $2\tau_{j+1} \le \tau_j/2$ by (7.7). The parameter choice (1.14) gives $L_{j+1}/L_j = K_{j+2} \ge 640$, so

$$\varepsilon_j = (L_j/L_{j+1})^2\varepsilon_{j-1} \le \varepsilon_{j-1}/32, \tag{7.9}$$

and $(\varepsilon\varepsilon_j)^{-1/2} = L_jL_{j+1} \le L_{j+1}^2$. In physical time the upper endpoint of the bracket in Proposition 2.8 lies at most τ_j after t_j. Indeed,

$$\frac{2(\varepsilon\varepsilon_j)^{-1/2}}{r} \le C\mu^{-1/8}L_{j+1}^2 \le \tau_j,$$

using $r \ge c\mu^{1/8}$ and $L_{j+1} \ge \log N$. Take t_{j+1} to be the first crossing in that interval. Its normalized time is at least $\tau_m \ge 2/\varepsilon > 1$, so $0 < t_{j+1} - t_j \le \tau_j$ and $\widetilde{\chi} = 1$ on $[t_{j+1}, S_j]$.

At this crossing (2.23) bounds $\sigma_j(t_{j+1})$ above and below by absolute multiples of $\lambda^{1/4}L_jL_{j+1}$, since $\max_{[t_j,S_j]}|\mathbf{a}_j(t,0)| = \lambda^{-3/4}$. As $1 \le L_jL_{j+1} \le L_{j+1}^2$, this gives the strength bounds in (7.3). Equation (2.19) applies at the same time and gives

$$\big|\log\big(\nu_j(t_{j+1})/\nu_{j-1}(t_j)\big)\big| \le 2\varepsilon_{j-1}.$$

Summing this inequality over the successive crossings from $\nu_1(t_2) = 2\pi$ and using $\varepsilon_i \le 32^{1-i}\varepsilon_1$ from (7.9) gives $|\log(\nu_j(t_{j+1})/(2\pi))| \le 4\varepsilon_1 \le 4(\log N)^{-2}$. Thus $\pi \le \nu_j(t_{j+1}) \le 4\pi$ for large N, completing (7.3).

On the interval where the cutoff changes, (6.8) gives $\gamma \le e^{3-L_j} = e^3\lambda_{j+1}^{-4}$. Since $|\widetilde{\chi}'| \le Cr$ and $Ce^3r \le \mu$ for large N by item 2,

$$\|\widetilde{\chi}'\gamma\|_\infty \le \mu\lambda_{j+1}^{-4}. \tag{7.10}$$

The same estimate holds for the first wave since $\widetilde{\chi}_1' \equiv 0$.

(4) *The expansion and its truncation.* Items 1 and 2 give the parent and wave assumptions of Subsection 5.1. By (7.8), $C_{\mathrm{step}} = \Gamma_P\mathsf{D}^b \le \mu^{10b} \le \lambda^{1/32}$, so (5.43) holds. We can therefore apply the expansion and truncation estimates of Section 5.

The new residual must be small at the frequency λ_{j+1} of the next wave. For the expansion remainder, the depth (7.2) gives $\lambda^{-2P/3} = \lambda^{-4K_{j+1}} = \lambda_{j+1}^{-4}$. For the time cutoff, item 3 gives the bound (7.10), with the same decay up to a factor μ. Thus (5.48) and $C_{\mathrm{step}}\mu \le \lambda^{1/16}$ give

$$\|R_V\|_{C_tC^{1,1/2}} \le 2\lambda^{2+1/16}\lambda_{j+1}^{-4} \le \lambda_{j+1}^{-7/2}. \tag{7.11}$$

To estimate the Fourier tail in (5.55), we need $\|V\|_{C_tC^{P,1/2}}$. For the parent $U = U_{j-1}$, (T1), (T2) and the Bernstein bound (B.1) give $\|U\|_{C_tC^{P,1/2}} \le C_P\mu^{5(P+1)+1} \le \lambda^{P+2}$, since $C_P \le \mu$ and $\lambda = \mu^{K_j}$ with $K_j \ge 6$. For the wave, (7.8) bounds the factor $C_{P,P}\mathsf{D}^{b(P+2)}$ in (5.48) by $\lambda^{(P+2)/32}$. The larger of the two resulting exponents for the oscillation and the

mean is $3P/2 + 1/4 + (P+2)/32$. This gives $\|W^\natural\|_{C_tC^{P,1/2}} \le \lambda^{2(P+1)}$ for large N. Hence $1 + \|V\|_{C_tC^{P,1/2}} \le 3\lambda^{2(P+1)}$. Inserting this with $\Lambda = \lambda^5$ and $9\mu \le \lambda$ gives

$$\mu\varepsilon_\Lambda \le 9\mu\lambda^{5(2-P)+4(P+1)} \le \lambda^{15-P} = \lambda^{15}\lambda_{j+1}^{-6} \le \lambda_{j+1}^{-5}. \tag{7.12}$$

In particular $C_{2,P}\varepsilon_\Lambda \le \lambda_{j+1}^{-5}$.

(5) *The new velocity.* Take $U_j = S_{\Lambda_j}V$ in Proposition 5.11. The proposition gives (T1), and (5.56) with (7.11) and (7.12) gives

$$\|R_j\|_{C_tL^\infty} \le C\lambda_{j+1}^{-7/2} + \lambda_{j+1}^{-5} \le \lambda_{j+1}^{-3}, \qquad \|v_\Lambda\|_{C_tC^2} \le \lambda_{j+1}^{-5}.$$

Since $U_j = V + v_\Lambda$ and $\lambda_{j+1}^{-5} \le \lambda^{-5/2}$, the divergence-free field v_Λ satisfies the hypothesis of Lemma 5.10. It follows that $\sup|(D_aX_j)^{\pm1}| \le C_{\mathrm{step}}\lambda^{1/4} \le \lambda$.

By Proposition 5.4, W_{osc} is supported in $|a| \le 2\ell_*$ in label coordinates. The support of $O_j = W_{\mathrm{osc}}^\natural$ lies in the image of this ball under the parent flow. The flow fixes the origin, so (5.1) and $\ell_* \le \mathsf{D}^{-3}$ place its Eulerian support in the ball of radius $2\mathsf{D}\ell_* \le 2\mathsf{D}^{-2} \le \mu^{-1}$. For $j \ge 2$, O_j vanishes before t_j by the same proposition. It vanishes also at t_j, since $\widetilde{\chi}(t_j) = 0$ and the higher oscillatory profiles have zero data there by Subsection 5.4. By (5.47) and (7.8), $\|O_j\|_{C_tL^\infty} \le \lambda^{1/32}\lambda^{-5/4} \le \lambda^{-1}$.

The rest of the velocity increment can extend across the torus. Since $V = U_{j-1} + W^\natural$,

$$U_j - U_{j-1} - O_j = \bar{W}^\natural + v_\Lambda.$$

For $0 \le m \le q_j$, the coefficient $C_{m,P}\mathsf{D}^{b(m+2)}$ in (5.48) is at most $\mu^{10b(m+2)} \le \lambda^{1/16}$, since $K_j = 320b(q_j+1)$. Thus $\|\bar{W}^\natural\|_{C_tC^{m,1/2}} \le \lambda^{1/16-5/2} \le \lambda^{-3/2}$. For the Fourier tail, (B.2) at order P and (7.12) give

$$\|v_\Lambda\|_{C_tC^{m,1/2}} \le \mu\Lambda_j^{m-P}\|V\|_{C_tC^{P,1/2}} \le \Lambda_j^{m-2}\mu\varepsilon_\Lambda \le \lambda^{5q_j}\lambda_{j+1}^{-5} \le \lambda_{j+1}^{-4},$$

where $K_{j+1} \ge K_j \ge 5q_j$. The sum of the mean and tail bounds is at most λ^{-1}, proving (7.4). In particular $\|U_j\|_{C_tL^\infty} \le \mu + 2\lambda^{-1} \le \lambda$ by (T2) for the parent. This completes (T2) for U_j.

(6) *Pressure and the central velocity gradient.* For the pressure increment, the leading term in (5.57) satisfies $C_{\mathrm{step}}\lambda^{-1/4} \le \lambda^{-7/32} \le \mu^{-2}$. By (7.11) and (7.12), the other two terms are at most $C\lambda_{j+1}^{-7/2} + \lambda_{j+1}^{-5}$. Hence $D^2\pi_j \le D^2\pi_{j-1} + \mu^{-1}\,\mathrm{Id}$ on $[0, S_j]$ for large N. Since $\mu^{-1} = \lambda_{j-1}^{-1}$ and $S_j \le S_{j-1}$, adding this increment to (T3) for the parent proves (T3) at step j.

At the origin truncation adds $\nabla v_\Lambda(t,0)$ to the gradient in (5.50). Thus the new error is $E_{c,j} = E_c + \nabla v_\Lambda(t,0)$ with $|E_{c,j}| \le \lambda^{-1/3} + \lambda_{j+1}^{-5} \le \lambda^{-1/4}$. From t_{j+1} onward the cutoff equals one, so the full rank-one term is present for the next wave. With the coupling and strength bounds from items 2 and 3, this proves (T4) and closes the induction.

The activation times increase strictly from $t_2 = 0$ onward. By (T4) their successive differences are bounded by τ_j, so (7.7) gives a finite limit $T_\infty \ge t_3 > 0$. For $j \ge 1$,

$$0 < T_\infty - t_j \le \sum_{p \ge j}\tau_p \le \tfrac{4}{3}\tau_j < S_j - t_j.$$

Thus $T_\infty < S_j$ for $j \ge 1$, and $T_\infty < S_0$ since $S_1 \le S_0$.

8. The limit and its terminal behavior

On the common interval $[0, T_\infty]$ of Theorem 7.1, (7.4) gives $\|U_j - U_{j-1}\|_{C_tL^\infty} \le 2\lambda_j^{-1}$. The sum converges, so the velocities converge uniformly to a continuous field w on $[0,T_\infty] \times \mathbb{T}^3$. For convergence of derivatives we use that the oscillations vanish before activation and have supports shrinking to the origin.

8.1. **Convergence before the accumulation time.**

Proposition 8.1. *The data $U_j(0)$ converge in every $C^{m,1/2}$ to an odd divergence-free field u_0. In particular, $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$. Let u be the maximal smooth solution of the Euler equations from u_0, with maximal time $T_* > 0$, supplied by Section 1. Then $T_* \geq T_\infty$ and, for every $T < T_\infty$ and integer $m \geq 0$, $U_j \to u$ in $C([0,T];C^{m,1/2})$. The solution is smooth and odd on $[0,T_\infty)\times\mathbb{T}^3$.*

Proof. Fix $T < T_\infty$ and $m \geq 1$. Since $t_j \uparrow T_\infty$ and $q_j \uparrow \infty$, we can choose j_0 with $T \leq t_{j_0}$ and $q_{j_0} \geq m$. For $j > j_0$ the oscillation O_j vanishes on $[0,T]$. Thus (7.4) gives, for $\ell > n \geq j_0$,

$$\|U_\ell - U_n\|_{C([0,T];C^{m,1/2})} \leq \sum_{j>n} \lambda_j^{-1} \leq 2\lambda_{n+1}^{-1}.$$

The limit in each of these norms is w. In particular $u_0 = w(0)$ is smooth, and w is odd and divergence free on $[0,T]$.

Integrating (7.5) gives

$$U_j(t) = U_j(0) - \int_0^t \mathbb{P}\operatorname{div}(U_j \otimes U_j)\,\mathrm{d}t' + \int_0^t R_j\,\mathrm{d}t'.$$

By (T2), the last integral has supremum norm at most $T\lambda_{j+1}^{-3}$. Convergence in $C([0,T];C^{1,1/2})$, the product estimate (5.59) and boundedness of $\mathbb{P}$ on $C^{0,1/2}$ give uniform convergence of the projected nonlinear term. Hence $\partial_t w = -\mathbb{P}\operatorname{div}(w\otimes w)$. The right-hand side is continuous in every spatial Hölder norm, so differentiating the equation gives joint smoothness. Uniqueness identifies w with u wherever both exist. Since w exists for every $T < T_\infty$, maximality gives $T_* \geq T_\infty$. □

8.2. **Velocity, central vorticity and the maximal time.**

Proposition 8.2. *The solution has bounded velocity on $[0,T_\infty)$, and $|\operatorname{curl} u(t,0)| \longrightarrow \infty$ as $t \uparrow T_\infty$.*

Proof. The continuous field w on $[0,T_\infty]\times\mathbb{T}^3$ is bounded, and Proposition 8.1 identifies it with u before T_∞.

For the vorticity, fix a large j and consider the whole interval $[t_j, t_{j+1}]$. The preceding wave is fully active there. Applying (T4) of Theorem 7.1 at steps $j-1$ and j gives, at the origin,

$$\nabla U_j = \nabla U_{j-2} + \mathbf{a}_{j-1}\otimes\kappa_{j-1} + \widetilde{\chi}_j\,\mathbf{a}_j\otimes\kappa_j + E_{c,j-1} + E_{c,j}.$$

A gradient $\mathbf{a}\otimes\kappa$ contributes $\kappa\times\mathbf{a}$ to vorticity. Projecting $\operatorname{curl} U_j$ along κ_j therefore removes the leading curl of wave j. The preceding wave contributes $\kappa_{j-1}\times\mathbf{a}_{j-1} = \sigma_{j-1}E_3$ in its own frame. By (2.3) and Lemma 2.4, the E_3 component of κ_j/λ_j is at least $1/2$ and its length is at most $C(1+\tau^2)$ at normalized time τ. The normalized length of $[t_j, S_j]$ is at most CL_{j+1}^4 by item 2 of the proof of Theorem 7.1. Thus

$$\frac{\kappa_j}{|\kappa_j|}\cdot(\kappa_{j-1}\times\mathbf{a}_{j-1}) \geq c\sigma_{j-1}L_{j+1}^{-8}, \qquad \frac{\kappa_j}{|\kappa_j|}\cdot(\kappa_j\times\mathbf{a}_j) = 0. \tag{8.1}$$

The preceding wave retains its strength throughout $[t_j, t_{j+1}]$. Put $\mu = \lambda_{j-1}$. At t_j, (7.3) gives $\sigma_{j-1} \geq c\mu^{1/4}$. By (2.2), $|\partial_t \log\sigma_{j-1}| \leq |\nabla U_{j-2}(t,0)| \leq \lambda_{j-2}^9 \leq \mu^{1/32}$, using (7.6) and item 2 above. Since the interval has length at most $\tau_j \leq \mu^{-1/16}$ by (T4) and (7.7), $\log\sigma_{j-1}$ changes by at most one. Hence $\sigma_{j-1} \geq c\mu^{1/4}$ on the whole interval. Since $L_{j+1}^8 \leq (\log\mu)^{16} \leq \mu^{1/8}$ for large j by (1.15), the projected curl of the preceding wave in (8.1) is at least $c\mu^{1/8}$. The older gradient contributes at most $C\lambda_{j-2}^9$ to the curl and the two central errors are at most $2\mu^{-1/4}$ in norm. Consequently

$$|\operatorname{curl} U_j(t,0)| \geq c\lambda_{j-1}^{1/8} - C\lambda_{j-2}^9 - C, \tag{8.2}$$

which tends to infinity uniformly on $[t_j, t_{j+1}]$.

For $\ell > j$ the oscillation O_ℓ vanishes throughout $[t_j, t_{j+1}]$. Applying (7.4) to the remaining increments at order one and taking the limit by Proposition 8.1 gives

$$\sup_{t_j \le t \le t_{j+1}} |\operatorname{curl}(u - U_j)(t,0)| \le C \sum_{\ell > j} \lambda_\ell^{-1}.$$

This tail tends to zero. The successive intervals cover $[t_j, T_\infty)$, so (8.2) gives $|\operatorname{curl} u(t,0)| \to \infty$ as $t \uparrow T_\infty$. □

Corollary 8.3. *The maximal existence time is $T_* = T_\infty \in (0,\infty)$, and both integrals in (1.2) diverge.*

Proof. By Proposition 8.1, $T_* \ge T_\infty$. If $T_* > T_\infty$, then smoothness on the compact interval $[0, T_\infty]$ would bound $\operatorname{curl} u(t,0)$, contrary to Proposition 8.2. Hence $T_* = T_\infty$, which is positive and finite by Theorem 7.1. The initial data are smooth, so the periodic Beale–Kato–Majda criterion [9], in the formulation of [11, Theorem 1], implies $\int_0^{T_*} \|\operatorname{curl} u(t)\|_\infty \, \mathrm{d}t = \infty$. The gradient integral also diverges, since $\|\operatorname{curl} u\|_\infty \le C \|\nabla u\|_\infty$. □

8.3. **The velocity away from the origin at the singular time.**

Proposition 8.4. *For every compact $K \subset \mathbb{T}^3 \setminus \{0\}$ and every integer $m \ge 0$, both $u(t)$ and $\partial_t u(t)$ converge in $C^m(K)$ as $t \uparrow T_\infty$.*

Proof. Fix K and m as in the statement. The support of O_j lies in the ball of radius λ_{j-1}^{-1} about the origin throughout $[0, T_\infty]$. For large j the ball is disjoint from K and $q_j \ge m$, so (7.4) gives $\|U_j - U_{j-1}\|_{C([0,T_\infty]; C^m(K))} \le \lambda_j^{-1}$. The increments are summable in this norm, hence $u(t) \to w(T_\infty)$ in $C^m(K)$.

The time derivative also contains the pressure gradient, which depends on the velocity on the whole torus. Choose a smooth cutoff χ equal to one near K and zero near the origin. On $[0, T_\infty)$ we have

$$\partial_t u = -\mathbb{P} \operatorname{div}(\chi u \otimes u) - \mathbb{P} \operatorname{div}\big((1-\chi) u \otimes u\big).$$

Applying the spatial convergence just proved on $\operatorname{supp} \chi$ through order $m+2$ gives convergence of $\chi u \otimes u$ in $C^{m+1,1/2}$ on the torus. Since $\mathbb{P} \operatorname{div}$ maps $C^{m+1,1/2}$ boundedly to $C^{m,1/2}$, the first term converges in $C^m(K)$.

For the second term, $\operatorname{supp}(1-\chi)$ is a positive distance from K. Only the pressure part $-\nabla \Delta^{-1} \operatorname{div} \operatorname{div}$ acts across this distance. Its kernel is smooth off the diagonal, so for any bounded tensor field A,

$$\|\mathbb{P} \operatorname{div}((1-\chi) A)\|_{C^m(K)} \le C_{m,K,\chi} \|A\|_{L^1}.$$

Since $u(t) \otimes u(t) \to w(T_\infty) \otimes w(T_\infty)$ uniformly on the torus, the difference tends to zero in L^1. The estimate therefore gives convergence of the second term in $C^m(K)$. □

Propositions 8.1, 8.2 and 8.4 and Corollary 8.3 prove Theorem 1.1.

Appendix A. The Hölder inverse

Subsection 4.3 reduces the choice of initial velocity to the endpoint problem (4.9). We solve it on the normalized interval $I = [0,1]$ and estimate the velocity in Hölder norm. Write $\|f\|_k = \sup_{t \in I} \|f(t)\|_{C^{k,1/2}}$. Constants C, p depend only on the dimension.

Let $K(t,a)$ be symmetric and $\Omega(t,a)$ skew-symmetric, continuous in time with values in $C^{1,1/2}$. Set

$$(Jf)(t) = \int_0^t f(s) \, \mathrm{d}s.$$

Assume, for some $\kappa, \mu > 0$,

$$K \geq \kappa I, \tag{A.1}$$

$$\int_I \big(K(t,a)z_t \cdot z_t + \Omega(t,a)z_t \cdot z\big)\,\mathrm{d}t \geq \mu \int_I |z_t|^2\,\mathrm{d}t \quad (a \in \mathbb{T}^3,\ z \in H^1_0(I;\mathbb{R}^3)). \tag{A.2}$$

Put

$$Q = 2 + \kappa^{-1} + \mu^{-1} + \|K\|_1 + \|\Omega\|_1 .$$

Theorem A.1. *Under (A.1)–(A.2), let $L \in C(I;C^{0,1/2})$. There are unique $z \in C^1(I;C^{0,1/2})$, $c \in C^{0,1/2}$ divergence free and $\nabla\phi \in C(I;C^{0,1/2})$ satisfying*

$$Kz_t - J(\Omega z_t) + L = c + \nabla\phi, \qquad \operatorname{div} z = 0, \qquad z(0) = z(1) = 0.$$

They obey

$$\|z_t\|_0 \leq CQ^p \|L\|_0 . \tag{A.3}$$

If K, Ω and L are smooth on $I \times \mathbb{T}^3$, then so are z, c and $\nabla\phi$.

Write $v = z_t$ and $z = Jv$. The terminal condition requires v to have zero time mean at each label. Let $\mathcal{Q}$ subtract this mean. The spatial Leray projection $\mathbb{P}$ removes the pressure gradient, and $\mathcal{Q}$ then removes the time-independent field c. The two projections commute. With $\mathcal{R} = \mathbb{P}\mathcal{Q}$, the equation for the solenoidal velocity becomes

$$\mathcal{R}Bv = -\mathcal{R}L, \qquad v = \mathcal{R}v, \qquad Bv = Kv - J(\Omega v).$$

For a path of zero time mean $w \in H = L^2(I;\mathbb{R}^3)$, we have $Jw \in H^1_0(I;\mathbb{R}^3)$ and $J^*w(t) = \int_t^1 w(s)\,\mathrm{d}s = -Jw(t)$. Hence at each label a,

$$\langle B(a)w, w\rangle_H = \int_I \big(Kw \cdot w + \Omega w \cdot Jw\big)\,\mathrm{d}t \geq \mu \|w\|_H^2$$

by (A.2) applied to Jw. This coercivity controls v in $C_a^{0,1/2}(L_t^2)$. To obtain continuity in time and (A.3), we also invert $v \mapsto \mathbb{P}(K(t)v)$ at each time using (A.1).

A.1. **Spatial freezing.** Both inversions have the form $\mathcal{R}(Bv) = f$ in the range of an orthogonal projection. Freezing B at a label makes this equation diagonal in the spatial Fourier modes, even when B acts on whole time paths.

Lemma A.2. *Let either $H = \mathbb{R}^3$, $\mathcal{R} = \mathbb{P}$, or $H = L^2(I;\mathbb{R}^3)$, $\mathcal{R} = \mathbb{P}\mathcal{Q}$, where $\mathcal{Q}v = v - \int_0^1 v(t)\,\mathrm{d}t$. Let $B \in C^{1,1/2}(\mathbb{T}^3;\mathcal{L}(H))$ and $\mu > 0$. Suppose*

$$\langle B(a)w, w\rangle_H \geq \mu \|w\|_H^2 \tag{A.4}$$

for every $a \in \mathbb{T}^3$ and every $w \in H$, with $\mathcal{Q}w = w$ in the second case. Then $\mathcal{R}B$ is an isomorphism on $\mathcal{R}C^{0,1/2}(\mathbb{T}^3;H)$, and

$$\|v\|_{C_a^{0,1/2}(H)} \leq C(2 + \mu^{-1} + \|B\|_{C_a^1\mathcal{L}(H)})^p \|\mathcal{R}Bv\|_{C_a^{0,1/2}(H)} . \tag{A.5}$$

If moreover $B \in C^{k,1/2}(\mathbb{T}^3;\mathcal{L}(H))$ for an integer $k \geq 1$, the same operator is an isomorphism on $\mathcal{R}C^{k,1/2}(\mathbb{T}^3;H)$.

Proof. For $\mathcal{R}v = v$, testing $f = \mathcal{R}(Bv)$ against v and using (A.4) gives

$$\mu \|v\|^2_{L^2_a(H)} \leq \langle Bv, v\rangle_{L^2_a(H)} = \langle f, v\rangle_{L^2_a(H)}.$$

This bound will control the lower-order term in the Hölder estimate.

Freeze $B_0 = B(a_0)$ and let $r(n)$ be the symbol of $\mathcal{R}$. For $n \neq 0$ the Leray symbol projects onto $n^\perp$, and at zero it is the identity. In the time-dependent case $r(n)$ also includes $\mathcal{Q}$. Extend the frozen operator to all of H by

$$A_0(n) = r(n)B_0 r(n) + I - r(n).$$

It acts as the identity on the orthogonal complement of $r(n)H$. Applying (A.4) to the real and imaginary parts of $r(n)w$ gives

$$\operatorname{Re}\langle A_0(n)w, w\rangle_H \geq \mu \|r(n)w\|_H^2 + \|(I - r(n))w\|_H^2 \geq \min(1,\mu)\|w\|_H^2 .$$

Lax–Milgram gives $\|A_0(n)^{-1}\| \leq \min(1,\mu)^{-1}$ [14, Section 6.2.1, Theorem 1].

The projection r is smooth on the unit sphere. The formula $D(A_0^{-1}) = -A_0^{-1}(DA_0)A_0^{-1}$ bounds its first five angular derivatives by a fixed polynomial in $2 + \mu^{-1} + \|B\|_\infty$. It follows that

$$\big\|A_0(D)^{-1} f\big\|_{C_a^{0,1/2}(H)} \leq C_0 \|f\|_{C_a^{0,1/2}(H)}, \qquad C_0 \leq C(2 + \mu^{-1} + \|B\|_\infty)^C. \tag{A.6}$$

Indeed, localize either symbol A_0^{-1} or $A_0^{-1}r$ to frequencies of size 2^j. Five integrations by parts on the rescaled annulus give the Euclidean kernel estimate

$$\|\nabla^d G_j(y)\|_{\mathcal{L}(H)} \leq C C_0\, 2^{(3+d)j}(1 + 2^j|y|)^{-5}, \qquad d = 0, 1.$$

For $d = 0$ the uniform L^1 bound persists under periodization. Applying the kernel to a slightly enlarged input block bounds the output block by $CC_0\, 2^{-j/2}\|f\|_{C_a^{0,1/2}(H)}$. Summing differences with one derivative on the low blocks and the supremum bound on the high blocks gives (A.6). The zero mode acts on the mean with norm at most $\min(1,\mu)^{-1}$.

Only near a_0 is the coefficient difference $B_0 - B$ small. Choose a cutoff χ equal to one on the ball of radius r about a_0, supported in the ball of radius $2r$, with $0 \leq \chi \leq 1$ and $|\nabla\chi| \leq C/r$. Splitting the source $(B_0 - B)v$ gives

$$\begin{aligned} v = {} & A_0(D)^{-1} f + A_0(D)^{-1}\mathcal{R}\{\chi(B_0 - B)v\} \\ & + A_0(D)^{-1}\mathcal{R}\{(1-\chi)(B_0 - B)v\}. \end{aligned} \tag{A.7}$$

The first term is controlled by (A.6). The coefficient $\chi(B_0 - B)$ has supremum norm at most $Cr\|B\|_{C^1}$ and Lipschitz constant at most $C\|B\|_{C^1}$. Its 1/2-Hölder norm is therefore at most $Cr^{1/2}\|B\|_{C^1}$. Put $Q_0 = 2 + \mu^{-1} + \|B\|_{C^1}$. Choosing $r = Q_0^{-M}$ with M sufficiently large bounds the second term in $C_a^{0,1/2}(H)$ by $\frac{1}{4}\|v\|_{C_a^{0,1/2}(H)}$.

For the last term the source vanishes on the ball of radius r. The periodized bounds for $d = 1$ sum to $CC_0(|y|^{-4} + 1)$ away from zero. At points within $r/2$ of a_0, integrating over torus distances $|y| \geq r/2$ bounds the gradient by $CC_0 r^{-1}\|B\|_\infty\|v\|_\infty$. For pairs of points at distance at most $r/2$, center the cutoff at one of them. The last term then contributes at most $CC_0 r^{-1/2}\|B\|_\infty\|v\|_\infty$ to the Hölder quotient. For other pairs use $Cr^{-1/2}\|v\|_\infty$ for the quotient of v. Absorbing the second term of (A.7) gives

$$\|v\|_{C_a^{0,1/2}(H)} \leq C_1\big(\|f\|_{C_a^{0,1/2}(H)} + \|v\|_{L_a^\infty(H)}\big), \qquad C_1 \leq CQ_0^C.$$

Averaging on a ball of radius ε^2 gives

$$\|v\|_\infty \leq \varepsilon\|v\|_{C_a^{0,1/2}(H)} + C\varepsilon^{-3}\|v\|_{L_a^2(H)} .$$

Taking $\varepsilon = (2C_1)^{-1}$ and using $\|v\|_{L_a^2(H)} \leq \mu^{-1}\|f\|_{L_a^2(H)}$ from coercivity proves (A.5).

To obtain existence from this estimate, set $B_s = (1-s)I + sB$. For $0 \leq s \leq 1$ its coercivity is at least $\min(1,\mu)$, so the estimate is uniform in s. When $B \in C^{k,1/2}$, apply it to $\partial^\nu v$ for $|\nu| \leq k$. Spatial derivatives commute with $\mathcal{R}$, so

$$\mathcal{R}B_s\partial^\nu v = \partial^\nu(\mathcal{R}B_s v) - \mathcal{R}[\partial^\nu, B_s]v.$$

Only lower derivatives of v occur in the commutator. Induction therefore gives $\|v\|_{C^{k,1/2}} \leq M_k \|\mathcal{R}B_s v\|_{C^{k,1/2}}$ uniformly in s. On $\mathcal{R}C^{k,1/2}$ the operator at $s=0$ is the identity. The uniform estimate and the Neumann series extend its inverse in finitely many steps to $s=1$, for $k=0$ and for every admissible $k \geq 1$. □

A.2. **Recovering the velocity at each time.**

Proof of Theorem A.1. On $H = L^2(I;\mathbb{R}^3)$ the coefficient $B = K - J\Omega$ satisfies $\|B\|_{C^1_a\mathcal{L}(H)} \leq C(\|K\|_1 + \|\Omega\|_1)$. With the coercivity obtained from (A.2), Lemma A.2 gives a unique solution $v = \mathbb{P}\mathcal{Q}v$ of $\mathbb{P}\mathcal{Q}(Bv+L)=0$, with

$$\|v\|_{C^{0,1/2}_a(L^2_t)} \leq CQ^p \|L\|_0 . \tag{A.8}$$

Taking the time mean recovers the field removed by $\mathcal{Q}$,

$$c = \mathbb{P}\int_0^1 (Bv+L)\,\mathrm{d}t.$$

Then $\mathbb{P}(Bv+L) = c$ in $L^2_t L^2_a$. In particular, for almost every time,

$$\mathbb{P}(Kv) = c + \mathbb{P}J(\Omega v) - \mathbb{P}L. \tag{A.9}$$

The right-hand side, denoted by $g(t)$, is already continuous in $C^{0,1/2}$. Indeed, (A.8) bounds c in $C^{0,1/2}$, and Cauchy–Schwarz on $[s,t]$ bounds the time increment of $J(\Omega v)$ in the same norm by $|t-s|^{1/2}\|\Omega v\|_{C^{0,1/2}_a(L^2_t)}$. Together with the hypothesis on L, this gives $\|g\|_0 \leq CQ^p\|L\|_0$.

For every t let $S(t)u = \mathbb{P}(K(t)u)$ on $\mathbb{P}C^{0,1/2}$. By (A.1), Lemma A.2 with $H=\mathbb{R}^3$ bounds $S(t)^{-1}$ uniformly by CQ^p. Since K is continuous in time, the resolvent identity gives continuity of $S(t)^{-1}$ in operator norm. Thus $\widetilde{v}(t) = S(t)^{-1}g(t)$ is continuous in $C^{0,1/2}$ and satisfies (A.3). For almost every t, the difference $u = v(t) - \widetilde{v}(t)$ is solenoidal in L^2_a and satisfies $\mathbb{P}(K(t)u) = 0$ by (A.9). Testing against u gives

$$\kappa\|u\|^2_{L^2_a} \leq \langle K(t)u, u\rangle_{L^2_a} = \langle \mathbb{P}(K(t)u), u\rangle_{L^2_a} = 0.$$

Thus v has the continuous representative $\widetilde{v}$ with the same zero time mean, so $z = Jv$ satisfies both endpoint conditions. At every time $\mathbb{P}(Bv+L-c)=0$, hence $Bv+L-c$ is a periodic gradient $\nabla\phi \in C(I;C^{0,1/2})$. Uniqueness of the projected solution v gives uniqueness of z, c and $\nabla\phi$.

For smooth data the higher-order assertion of Lemma A.2 first gives $v \in C^{k,1/2}_a(L^2_t)$, hence $c \in C^{k,1/2}$ and $J(\Omega v) \in C(I;C^{k,1/2})$. The fixed-time inverse then gives $v \in C(I;C^{k,1/2})$ for each k. Since K is smooth, $S(t)^{-1}$ is smooth in operator norm on $\mathbb{P}C^{k,1/2}$. In (A.9), $J(\Omega v)$ gains one time derivative from v. Induction gives all time derivatives of v in every $C^{k,1/2}$ and thus smoothness of z and $\nabla\phi$. □

Appendix B. The frequency cutoff

Fix a radial $\psi \in C^\infty_c(\mathbb{R}^3)$ with $0 \leq \psi \leq 1$, $\psi = 1$ on $|\zeta| \leq 1$ and $\operatorname{supp}\psi \subset \{|\zeta| \leq 2\}$. For $\Lambda \geq 1$ let S_Λ multiply the spatial Fourier mode $e^{2\pi i\xi\cdot x}$, $\xi \in \mathbb{Z}^3$, by $\psi(\xi/\Lambda)$. Subsection 5.7 takes $u = S_\Lambda V$ as the next parent. The frequencies of u are at most 2Λ, and those of $u\cdot\nabla u$ are at most 4Λ. Thus (B.1) bounds derivatives of both u and its residual, while (B.2) controls the error $u - V$.

Lemma B.1. *S_Λ commutes with derivatives, preserves parity and divergence freeness, and maps into fields supported on frequencies at most 2Λ. Its operator norms on L^∞ and on $C^{0,1/2}$ are bounded in terms of ψ alone. For every integer $m \geq 0$ and every normed-space-valued f supported on frequencies at most 4Λ,*

$$\sum_{|\beta|=m} \left\|\partial^\beta f\right\|_\infty \leq C_m \Lambda^m \|f\|_\infty . \tag{B.1}$$

For all integers $0 \le s \le M$ and every $f \in C^{M,1/2}$,

$$\|(\mathrm{Id} - S_\Lambda) f\|_{C^{s,1/2}} \le C_M \Lambda^{s-M} \|f\|_{C^{M,1/2}} . \tag{B.2}$$

The constants C_m and C_M depend only on their index and on ψ.

Proof. The symbol is scalar, even and independent of time, so the cutoff preserves parity and divergence freeness and commutes with differentiation. Its support gives the frequency bound. For the norm estimates we use the convolution kernel as in the Euclidean Bernstein inequalities [55, Appendix A, (A.2)–(A.6)]. Let $K = \check{\psi}$, a Schwartz function, and set $K_\Lambda(y) = \Lambda^3 K(\Lambda y)$. To compute $S_\Lambda f$, convolve the periodic extension of f to $\mathbb{R}^3$ with K_Λ. Since $\|K_\Lambda\|_{L^1} = \|K\|_{L^1}$, applying convolution to f and to its spatial increments gives the two operator bounds. If f has frequencies at most 4Λ, then $f = S_{4\Lambda} f$. The identity $\|\partial^\beta K_{4\Lambda}\|_{L^1} = (4\Lambda)^{|\beta|} \|\partial^\beta K\|_{L^1}$ gives (B.1) also for normed-space-valued f.

For (B.2) put $k = M - s$. When $k = 0$, apply the $C^{0,1/2}$ bound to the derivatives of f through order s. For $k \ge 1$, the kernel has integral one and all its positive-order moments vanish because ψ is constant near zero. Thus in $f - S_\Lambda f$ only the remainder after expanding $f(x-y)$ at $y = 0$ through degree $k-1$ contributes. Its $C^{s,1/2}$ norm in x is bounded by $C_M |y|^k \|f\|_{C^{M,1/2}}$, by Taylor's formula for each derivative through order s and its spatial increments. Integrating this bound against $|K_\Lambda(y)|$ gives (B.2), since $\int_{\mathbb{R}^3} |y|^k |K_\Lambda(y)| \, \mathrm{d}y = \Lambda^{-k} \int_{\mathbb{R}^3} |y|^k |K(y)| \, \mathrm{d}y$. □

Appendix C. The parameter inequalities

Proof of Lemma 1.3. The current order q_j is always admissible in the maximum (1.14). Indeed, $\mathcal{B}(q_1) = \mathcal{B}(1) \le N = \lambda_0$, and for $j \ge 2$ the preceding choice gives $\mathcal{B}(q_j) \le \lambda_{j-2} \le \lambda_{j-1}$. Thus $q_{j+1} = q_j$ until λ_{j-1} reaches $\mathcal{B}(q_j + 1)$, at which point the order increases by one. Since $N > 1$ and $K_j \ge 2$, the frequencies tend to infinity. They eventually exceed every fixed threshold, so $q_j \to \infty$.

The gain L_{j+1} involves three successive frequency steps. By (1.14),

$$L_{j+1} = 4 \log \lambda_{j+2} = 4 K_j K_{j+1} K_{j+2} \log \lambda_{j-1}.$$

Each exponent is at most $K(q_{j+1} + 1)$, since the orders increase by at most one. Thus (1.15) follows from (1.13) and the choice of q_{j+1}, which give

$$4 K_j K_{j+1} K_{j+2} \le 4 K (q_{j+1} + 1)^3 \le \log \mathcal{B}(q_{j+1}) \le \log \lambda_{j-1}.$$ □

Appendix D. Finite norms and coefficient estimates

In Section 5 we estimate the mean and the oscillatory corrections at every derivative order. We measure label derivatives on the envelope radius ℓ_* and phase derivatives on the profile width h, as chosen in Subsection 5.1. With these weights the power of the parent bound D in the inverse estimates is independent of the derivative order.

Write $|f|_m = \sum_{|\beta|=m} \sup |\partial^\beta f|$ for the spatial derivatives of order m, taken in the label a or the position x as appropriate. The supremum includes all variables. For a field $f(t, a, s)$ of period one in s and an integer $n \ge 0$, set

$$\|f\|_{[n]} = \sum_{m+j \le n} \ell_*^m h^j |\partial_s^j f|_m, \qquad \|f\|_{[n],\gamma} = \|f/\gamma\|_{[n]}. \tag{D.1}$$

The weighted norm is taken on $[t_0, T]$, where $\gamma(t) > 0$ measures the amplitude of the seed at the origin. All norms may be restricted to a closed subinterval of their time domain.

For a phase-independent field $c(t, a)$, only the terms with $j = 0$ occur in (D.1). The mean inverse of Theorem 4.2 uses spatial Hölder norms, so we put

$$|c|_{m,1/2} = \sum_{|\beta|=m} \sup_t \left\| \partial_a^\beta c(t) \right\|_{C^{0,1/2}}, \qquad \|c\|_{[n],1/2} = \sum_{m \le n} \ell_*^m |c|_{m,1/2}.$$

To pass from supremum bounds to this norm costs one derivative and a factor $\ell_*^{-1/2}$. Indeed, bounding $|v(x)-v(y)|$ by both $2\|v\|_\infty$ and $\|\nabla v\|_\infty|x-y|$ gives $[v]_{C^{0,1/2}} \le (2\|v\|_\infty\|\nabla v\|_\infty)^{1/2}$. Applying this to $\partial_a^\beta c$ and using $\ell_* \le 1$ gives

$$\ell_*^m|c|_{m,1/2} \le C\ell_*^{-1/2}\big(\ell_*^m|c|_m+\ell_*^{m+1}|c|_{m+1}\big), \qquad \text{hence} \quad \|c\|_{[n]} \le \|c\|_{[n],1/2} \le C\ell_*^{-1/2}\|c\|_{[n+1]}. \tag{D.2}$$

Lemma D.1. *For every $n \ge 0$ and fields f, g,*

$$\sum_i \|\partial_{a_i} f\|_{[n],\gamma} \le C\ell_*^{-1}\|f\|_{[n+1],\gamma}, \qquad \|\partial_s f\|_{[n],\gamma} \le h^{-1}\|f\|_{[n+1],\gamma}, \tag{D.3}$$

$$\|fg\|_{[n],\gamma} \le C_n\|f\|_{[n]}\|g\|_{[n],\gamma}. \tag{D.4}$$

With $\langle f\rangle_s$ the phase average,

$$\|\langle f\rangle_s\|_{[n],\gamma} \le \|f\|_{[n],\gamma}, \qquad \|f-\langle f\rangle_s\|_{[n],\gamma} \le 2\|f\|_{[n],\gamma}, \qquad \|\partial_s^{-1}f\|_{[n],\gamma} \le C\|f\|_{[n],\gamma} \quad (\langle f\rangle_s = 0). \tag{D.5}$$

For phase-independent fields, (D.4) also holds with $\|\cdot\|_{[n],1/2}$ in place of $\|\cdot\|_{[n]}$ and $\gamma \equiv 1$.

Proof. Each label derivative of order $m+1$ occurs at most three times in the sum over i in (D.3). The phase derivative shifts j by one in (D.1). For (D.4), apply Leibniz' rule to $f(g/\gamma)$, using that γ depends only on time. For the Hölder version use the product inequality in $C^{0,1/2}$.

Phase averaging has supremum norm at most one. For a zero-mean f, subtracting the average of $\int_0^s f(t,a,\sigma)\,\mathrm{d}\sigma$ gives its zero-mean periodic primitive, of supremum norm at most $2\sup|f|$. The phase operators commute with label and phase derivatives, so these bounds give (D.5) at every order. □

The coefficients in Subsections 5.3 and 5.4 are independent of the phase. Their label derivatives satisfy bounds of the form

$$|c|_m \le C_m\mathsf{D}^{C(m+1)} \qquad (m \ge 0). \tag{D.6}$$

The absolute exponent C may depend on the coefficient. The next lemma allows us to multiply and invert such coefficients, and shows how much of the factor ℓ_*^m is available when at least one derivative falls on a coefficient.

Lemma D.2. *Let c, d be smooth phase-independent matrix fields that satisfy (D.6).*

(a) The product cd satisfies (D.6).

(b) If c is invertible and $\sup_{t,a}|c^{-1}|$ is at most a power of D with an absolute exponent, then c^{-1} satisfies (D.6).

(c) If $\ell_* \le \mathsf{D}^{-C}$ with the exponent C of (D.6) for c, then $\|c\|_{[n]} \le C_n\mathsf{D}^C$ for every $n \ge 0$, and $\ell_*^m|c|_m \le C_m\mathsf{D}^{2C}\ell_*$ for every $m \ge 1$.

Proof. For (a) and (b) take C large enough for the bounds on c, d and, in (b), on $\sup|c^{-1}|$. Leibniz' rule gives $|cd|_m \le C_m\mathsf{D}^{C(m+2)}$, proving (a) with exponent $2C$. For $e = c^{-1}$, differentiating $ce = I$ gives

$$|e|_m \le C_m\mathsf{D}^C\sum_{i=1}^m |c|_i|e|_{m-i} \qquad (m \ge 1).$$

Starting from $|e|_0 \le \mathsf{D}^C$, induction gives $|e|_m \le C_m\mathsf{D}^{C(3m+1)}$, proving (b) with exponent $3C$. For (c) use the exponent C for c in (D.6). Then $\ell_*^m|c|_m \le C_m\mathsf{D}^C(\ell_*\mathsf{D}^C)^m$. Since $\ell_*\mathsf{D}^C \le 1$, summing gives the first bound, and retaining one factor $\ell_*\mathsf{D}^C$ for $m \ge 1$ gives the second. □

For the mean we differentiate the integrated equation (4.9) in the unweighted Hölder norm. For the oscillations we differentiate (5.18) in the γ-weighted norm. Its unknown takes values in a fixed $\mathbb{R}^2$, so label differentiation preserves the space of unknowns. In both equations a derivative falling

on a coefficient leaves fewer derivatives on the solution. The following estimate uses this fact to retain just one factor of the zero-order inverse bound.

Lemma D.3. *Consider a linear equation of the form*

$$\mathcal{L}v = \mathcal{L}_0 v + \sum_\nu T_\nu(c_\nu v) = F$$

with finitely many phase-independent coefficients $c_\nu(t,a)$. Assume that $\mathcal{L}_0$ and the linear operators T_ν commute with label and phase derivatives. The equation may hold modulo a fixed subspace, such as gradients. Assume that differentiation preserves this subspace and the linear constraints on v, and commutes with the maps giving any prescribed data. Write f for the forcing F together with these data, with the sum of their norms. Suppose that for every admissible source one has the estimate

$$\|v\|_{[0],\gamma} \le C_{\mathrm{low}} \|f\|_{[0],\gamma}.$$

Let b_ν be the operator norm of T_ν from $\|\cdot\|_{[0],\gamma}$ to the forcing norm. If

$$C_{\mathrm{low}} \sum_\nu b_\nu \ell_*^m |c_\nu|_m \le C_m \qquad (m \ge 1), \tag{D.7}$$

then every smooth solution satisfies, for $n \ge 0$,

$$\|v\|_{[n],\gamma} \le C_n C_{\mathrm{low}} \|f\|_{[n],\gamma}.$$

Here C_n depends only on n and the bounds C_m in (D.7) for $m \le n$. For phase-independent fields with $\gamma \equiv 1$, the same assertion holds using $\|\cdot\|_{[n],1/2}$ and $|c_\nu|_{m,1/2}$ throughout, with b_ν taken in the corresponding zero-order norms.

Proof. Phase derivatives commute with the equation. For a label multi-index β and phase order j, the differentiated forcing is

$$\mathcal{L}(\partial_a^\beta \partial_s^j v) = \partial_a^\beta \partial_s^j F - \sum_\nu \sum_{0<\beta' \le \beta} \binom{\beta}{\beta'} T_\nu\big((\partial_a^{\beta'} c_\nu)\partial_a^{\beta-\beta'}\partial_s^j v\big).$$

The zero-order estimate applies with the differentiated data. Multiply by $\ell_*^{|\beta|} h^j$ and sum over $|\beta| + j \le n$. A term with $i = |\beta'| \ge 1$ assigns ℓ_*^i to the coefficient and leaves a norm of order at most $n - i$ on v. Thus for $n \ge 1$,

$$\|v\|_{[n],\gamma} \le C_{\mathrm{low}} \|f\|_{[n],\gamma} + C_n \sum_{i=1}^{n} \Big(C_{\mathrm{low}} \sum_\nu b_\nu \ell_*^i |c_\nu|_i \Big) \|v\|_{[n-i],\gamma}.$$

By (D.7), the sum is bounded by $C_n \|v\|_{[n-1],\gamma}$. Induction from order zero gives the claim. For the Hölder norms the same argument uses the product inequality in $C^{0,1/2}$. □

In Section 5 the zero-order bounds come from Theorem 4.2 and (5.23). For $m \ge 1$, Lemma D.2(c) leaves a factor ℓ_* in $\ell_*^m |c_\nu|_m$. In the Hölder norm, (D.2) leaves a factor $\ell_*^{1/2}$, as in (5.13). In each case $C_{\mathrm{low}} \sum_\nu b_\nu$ is bounded by a fixed power of D. One choice $\ell_* = \mathsf{D}^{-A}$ with A sufficiently large therefore gives (D.7) at every order.

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